

# Viscosity Solutions of Fully Nonlinear Parabolic Path Dependent PDEs: Part II

Ibrahim EKREN\*      Nizar TOUZI†      Jianfeng ZHANG‡

October 2, 2012

## Abstract

In our previous paper [7], we have introduced a notion of viscosity solutions for path-dependent fully nonlinear PDEs, extending the semilinear case of [5], which satisfies a partial comparison result under standard Lipschitz-type assumptions. The main result of this paper provides a full wellposedness result under an additional assumption formulated on some partial differential equation defined locally by freezing the path. Namely, assuming further that such path-frozen standard PDEs satisfy the comparison principle and the Perron approach for existence, we prove that the path-dependent nonlinear PDE has a unique viscosity solution. Uniqueness is implied by a comparison result.

**Key words:** Path dependent PDEs, Second order Backward SDEs,  $G$ -martingales, viscosity solutions, comparison principle.

**AMS 2000 subject classifications:** 35D40, 35K10, 60H10, 60H30.

---

\*University of Southern California, Department of Mathematics, ekren@usc.edu.

†CMAP, Ecole Polytechnique Paris, nizar.touzi@polytechnique.edu. Research supported by the Chair *Financial Risks* of the *Risk Foundation* sponsored by Société Générale, and the Chair *Finance and Sustainable Development* sponsored by EDF and Calyon.

‡University of Southern California, Department of Mathematics, jianfenz@usc.edu. Research supported in part by NSF grant DMS 10-08873.

# 1 Introduction

Consider the fully nonlinear path-dependent parabolic partial differential equation defined on the space of continuous paths  $\Omega = \{\omega \in C^0([0, T], \mathbb{R}^d) : \omega_0 = 0\}$ :

$$\{-\partial_t u - G(\cdot, u, \partial_\omega u, \partial_{\omega\omega}^2 u)\}(t, \omega) = 0, \quad t < T, \quad \omega \in \Omega,$$

for some progressively measurable nonlinearity  $G : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ , where  $\mathbb{S}^d$  is the set of symmetric matrices of size  $d$  with real entries. Such equations were considered first by Lukoyanov [13] in the first order case, and discussed by Peng [17, 18].

A smooth process  $(t, \omega) \mapsto u(t, \omega)$  is defined as a progressively measurable process such that the Dupire [4] time-derivative  $\partial_t u$  exists and is continuous, and  $u$  is a semimartingale under a convenient family of nondominated singular measures with decomposition  $du = \partial_t u dt + \frac{1}{2} \partial_{\omega\omega}^2 u d\langle B \rangle + \partial_\omega u dB_t$ , where  $B$  denotes the canonical process.

When the nonlinearity  $G$  is semilinear, i.e. linear with respect to the  $\partial_{\omega\omega}^2 u$ -component, the theory of backward stochastic differential equations, started by Pardoux and Peng [16], provides a wellposedness result for (1.1) in a strong sense imposing the existence of the space gradient  $\partial_\omega u$  in a convenient functional space. For a special class of fully nonlinear maps  $G$ , a similar theory was developed by means of second order backward stochastic differential equations introduced by Cheridito, Soner, Touzi, and Victoir [1], Soner, Touzi and Zhang [20], and the closely related G-BSDEs of Hu, Ji, Peng and Song [9].

Our objective in this paper and the accompanying one [7] is to relax the strong requirement of existence of the space gradient in the theory of backward stochastic differential equations and its various extensions. To do this, we follow the classical theory of viscosity solutions of partial differential equations, introduced by Crandall and Lions [2], see [3] for an overview and [8] for a pedagogical presentation. The main difficulty is that the classical finite-dimensional theory of viscosity solutions is strongly based on the local compactness of the underlying space. Since the space of continuous paths does not satisfy this condition, our main contribution is to introduce a definition of viscosity solutions which circumvents this major difficulty. This is achieved in [5] and [7] by replacing the pointwise extremality in the standard definition of viscosity solutions by the corresponding extremality in the context of an optimal stopping problem under nonlinear expectation. In the fully nonlinear case, the delicate analysis of such optimal stopping problems is reported in our paper [6].

This paper is a continuation of [7], and contains the main wellposedness theory for the above path dependent PDE under convenient assumptions. Our starting point is the partial comparison result established in [7] which states, under fairly general Lipschitz-

type conditions on the nonlinearity  $G$ , that for any bounded viscosity subsolution  $u^1$  and supersolution  $u^2$  with  $u_T^1 \leq u_T^2$ , we have  $u^1 \leq u^2$  on  $[0, T] \times \Omega$ , provided that one of them is smooth. Then, similar to the approach of [5] in the semilinear case, we use the Perron approach to construct a viscosity solution on one hand, and to prove that the comparison result of bounded viscosity subsolutions and supersolutions holds true without the requirement that any one of them is smooth. While the corresponding wellposedness results in the semilinear context of [5] rely on the representation of the solution of (1.1) by means of a backward stochastic differential equation, we introduce a new argument in this paper which builds upon the local wellposedness of the path-frozen partial differential equation. The path-frozen PDE is defined locally on the finite dimensional space  $\mathbb{R}^d$ , so that the last wellposedness assumption is related to the standard theory of viscosity solutions as in [3].

Our wellposedness results cover path dependent PDEs which are not accessible in the existing literature on backward stochastic differential equations. For instance, the nonlinearity  $G$  does not need to fulfill the strong requirements of second order backward SDEs (e.g. convexity in the  $\partial_{\omega\omega}^2 u$ -component) Another example is the general class of backward stochastic partial differential equations, which appear naturally in many applications, see e.g. Ma, Yin and Zhang [14] on non-Markovian FBSDEs, and Oksendal, Sulem and Zhang [15] on stochastic control of SPDEs.

The rest of the paper is organized as follows. Section 2 introduces the general framework, and recalls the definition of viscosity solutions introduced in our accompanying paper [7]. The main results and the corresponding assumptions are reported in Section 3. To prepare for the proof, we start by providing in Section 4 a stronger partial comparison result extending that of [7]. Section 5 is devoted to the proof of the comparison result, which implies uniqueness. The existence result is proved in Section 6. Finally, Section 7 provides some sufficient conditions for our main assumption under which our wellposedness result is established.

## 2 Preliminaries

### 2.1 The canonical spaces

Let  $\Omega := \{\omega \in C([0, T], \mathbb{R}^d) : \omega_0 = \mathbf{0}\}$ , the set of continuous paths starting from the origin,  $B$  the canonical process,  $\mathbb{F}$  the filtration generated by  $B$ ,  $\mathbb{P}_0$  the Wiener measure, and  $\Lambda := [0, T] \times \Omega$ . Here and in the sequel, for notational simplicity, we use  $\mathbf{0}$  to denote

vectors or matrices with appropriate dimensions whose components are all equal to 0. Let  $\mathbb{S}^d$  denote the set of  $d \times d$  matrices, and

$$x \cdot x' := \sum_{i=1}^d x_i x'_i \text{ for any } x, x' \in \mathbb{R}^d, \quad \gamma : \gamma' := \text{Trace}[\gamma \gamma'] \text{ for any } \gamma, \gamma' \in \mathbb{S}^d.$$

We define a norm on  $\Omega$  and a metric on  $\Lambda$  as follows: for any  $(t, \omega), (t', \omega') \in \Lambda$ ,

$$\|\omega\|_t := \sup_{0 \leq s \leq t} |\omega_s|, \quad \mathbf{d}_\infty((t, \omega), (t', \omega')) := |t - t'| + \|\omega_{\cdot \wedge t} - \omega'_{\cdot \wedge t'}\|_T. \quad (2.1)$$

Then  $(\Omega, \|\cdot\|_T)$  and  $(\Lambda, \mathbf{d}_\infty)$  are complete metric spaces. We denote by  $C^0(\Lambda)$  (resp.  $UC(\Lambda)$ ) the collection of all continuous (resp. uniformly continuous) progressively measurable processes on  $\Lambda$  with respect to  $\mathbf{d}_\infty$ . The corresponding subspaces of bounded processes is denoted  $C_b^0(\Lambda)$  and  $UC_b(\Lambda)$ . Moreover, we denote by  $C^0(\Lambda, \mathbb{R}^d)$  the space of  $\mathbb{R}^d$ -valued processes whose components are in  $C^0(\Lambda)$ , and define other similar notations in the same spirit.

**Definition 2.1** By  $\underline{\mathcal{U}}$ , we denote the collection of all  $\mathbb{F}$ -progressively measurable processes  $u : \Lambda \rightarrow \mathbb{R}$  such that  $u$  is bounded from above and

- for all  $\omega \in \Omega$ ,  $t \mapsto u(t, \omega)$  is càdlàg with positive jumps,
- there exists a modulus of continuity function  $\rho$  such that for any  $(t, \omega), (t', \omega') \in \Lambda$ :

$$u(t, \omega) - u(t', \omega') \leq \rho\left(\mathbf{d}_\infty((t, \omega), (t', \omega'))\right) \text{ whenever } t \leq t'. \quad (2.2)$$

By  $\overline{\mathcal{U}}$  we denote the set of all processes  $u$  such that  $-u \in \underline{\mathcal{U}}$ .

The progressive measurability of  $u$  implies that  $u(t, \omega) = u(t, \omega_{\cdot \wedge t})$ , and it is clear that  $\underline{\mathcal{U}} \cap \overline{\mathcal{U}} = UC_b(\Lambda)$ . We also recall from [6] Remark 3.2 the redundancy in the above assumption that Condition (2.2) implies that  $X$  has left-limits and positive jumps.

We next introduce the shifted spaces. Let  $0 \leq s \leq t \leq T$ .

- Let  $\Omega^t := \{\omega \in C([t, T], \mathbb{R}^d) : \omega_t = \mathbf{0}\}$  be the shifted canonical space;  $B^t$  the shifted canonical process on  $\Omega^t$ ;  $\mathbb{F}^t$  the shifted filtration generated by  $B^t$ ,  $\mathbb{P}_0^t$  the Wiener measure on  $\Omega^t$ , and  $\Lambda^t := [t, T] \times \Omega^t$ .

- Define  $\|\cdot\|_s$  on  $\Omega^t$ ,  $\mathbf{d}_\infty$  on  $\Lambda^t$ , and  $C^0(\Lambda^t)$  etc. in the spirit of (2.1) and Definition 2.1.
- For  $\omega \in \Omega^s$  and  $\omega' \in \Omega^t$ , define the concatenation path  $\omega \otimes_t \omega' \in \Omega^s$  by:

$$(\omega \otimes_t \omega')(r) := \omega_r \mathbf{1}_{[s, t)}(r) + (\omega_t + \hat{\omega}'_r) \mathbf{1}_{[t, T]}(r), \quad \text{for all } r \in [s, T].$$

- Let  $s \in [0, T)$  and  $\omega \in \Omega^s$ . For an  $\mathcal{F}_T^s$ -measurable random variable  $\xi$ , an  $\mathbb{F}^s$ -progressively measurable process  $X$  on  $\Omega^s$ , and  $t \in (s, T]$ , define the shifted  $\mathcal{F}_T^t$ -measurable

random variable  $\xi^{t,\omega}$  and  $\mathbb{F}^t$ -progressively measurable process  $X^{t,\omega}$  on  $\Omega^t$  by:

$$\xi^{t,\omega}(\omega') := \xi(\omega \otimes_t \omega'), \quad X^{t,\omega}(\omega') := X(\omega \otimes_t \omega'), \quad \text{for all } \omega' \in \Omega^t.$$

It is clear that, for any  $(t, \omega) \in \Lambda$  and any  $u \in C^0(\Lambda)$ , we have  $u^{t,\omega} \in C^0(\Lambda^t)$ . Similar property holds for other spaces introduced above.

Finally, we denote by  $\mathcal{T}$  the set of  $\mathbb{F}$ -stopping times, and  $\mathcal{H} \subset \mathcal{T}$  the subset of those hitting times  $H$  of the form

$$H := \inf\{t : B_t \in O^c\} \wedge t_0 = \inf\{t : d(\omega_t, O^c) = 0\} \wedge t_0, \quad (2.3)$$

for some  $0 < t_0 \leq T$ , and some open and convex set  $O \subset \mathbb{R}^d$  containing  $\mathbf{0}$ . The set  $\mathcal{H}$  will be important for our optimal stopping problem, which is crucial for the comparison and the stability results. We note that  $H = t_0$  when  $O = \mathbb{R}^d$ . Moreover,

$$H > 0, H \text{ is lower semicontinuous, and } H_1 \wedge H_2 \in \mathcal{H} \text{ for any } H_1, H_2 \in \mathcal{H}.$$

Define  $\mathcal{T}^t$  and  $\mathcal{H}^t$  in the same spirit. For any  $\tau \in \mathcal{T}$  (resp.  $H \in \mathcal{H}$ ) and any  $(t, \omega) \in \Lambda$  such that  $t < \tau(\omega)$  (resp.  $t < H(\omega)$ ), it is clear that  $\tau^{t,\omega} \in \mathcal{T}^t$  (resp.  $H^{t,\omega} \in \mathcal{H}^t$ ).

## 2.2 Capacity and nonlinear expectation

For all  $L > 0$  and  $t \in [0, T]$ , let  $\mathcal{P}_L^t$  denote the set of probability measures  $\mathbb{P}$  on  $\Omega^t$  such that there exist  $\mathbb{F}^t$ -progressively measurable  $\mathbb{R}^d$ -valued processes  $\alpha^\mathbb{P}$ , an  $\mathbb{S}^d$ -valued process  $\beta^\mathbb{P}$ , and a  $d$ -dimensional  $\mathbb{P}$ -Brownian motion  $W^\mathbb{P}$  satisfying:

$$|\alpha^\mathbb{P}| \leq L, \quad 0 \leq \beta^\mathbb{P} \leq \sqrt{2L}I_d, \quad dB_t = \beta_t^\mathbb{P} dW_t^\mathbb{P} + \alpha_t^\mathbb{P} dt, \quad \mathbb{P}\text{-a.s.} \quad (2.4)$$

and we define  $\mathcal{P}_\infty^t := \bigcup_{L>0} \mathcal{P}_L^t$ . The set  $\mathcal{P}_L^t$  induces the following capacity:

$$\mathcal{C}_t^L[A] := \sup_{\mathbb{P} \in \mathcal{P}_L^t} \mathbb{P}[A], \quad \text{for all } A \in \mathcal{F}_T^t. \quad (2.5)$$

We denote by  $\mathbb{L}^1(\mathcal{F}_T^t, \mathcal{P}_L^t)$  the set of all  $\mathcal{F}_T^t$ -measurable r.v.  $\xi$  with  $\sup_{\mathbb{P} \in \mathcal{P}_L^t} \mathbb{E}^\mathbb{P}[|\xi|] < \infty$ . The following nonlinear expectation will play a crucial role:

$$\bar{\mathcal{E}}_t^L[\xi] = \sup_{\mathbb{P} \in \mathcal{P}_L^t} \mathbb{E}^\mathbb{P}[\xi] \quad \text{and} \quad \underline{\mathcal{E}}_t^L[\xi] = \inf_{\mathbb{P} \in \mathcal{P}_L^t} \mathbb{E}^\mathbb{P}[\xi] = -\bar{\mathcal{E}}_t^L[-\xi] \quad \text{for all } \xi \in \mathbb{L}^1(\mathcal{F}_T^t, \mathcal{P}_L^t). \quad (2.6)$$

**Definition 2.2** Let  $X$  be a progressively measurable process with  $X_t \in \mathbb{L}^1(\mathcal{F}_t, \mathcal{P}_L)$ . We say that  $X$  is an  $\bar{\mathcal{E}}^L$ -supermartingale (resp. submartingale, martingale) if, for any  $(t, \omega) \in \Lambda$  and any  $\tau \in \mathcal{T}^t$ ,  $\bar{\mathcal{E}}_t^L[X_\tau^{t,\omega}] \leq$  (resp.  $\geq, =$ )  $X_t(\omega)$ .

We now state the Snell envelope characterization of optimal stopping under the above nonlinear expectation operators. Given a bounded progressively measurable process  $X$ , consider the nonlinear optimal stopping problem

$$\overline{\mathcal{S}}_t^L[X](\omega) := \sup_{\tau \in \mathcal{T}^t} \overline{\mathcal{E}}_t^L[X_\tau^{t,\omega}] \quad \text{and} \quad \underline{\mathcal{S}}_t^L[X](\omega) := \inf_{\tau \in \mathcal{T}^t} \underline{\mathcal{E}}_t^L[X_\tau^{t,\omega}], \quad (t, \omega) \in \Lambda. \quad (2.7)$$

By definition, we have  $\overline{\mathcal{S}}^L[X] \geq X$  and  $\overline{\mathcal{S}}_T^L[X] = X_T$ .

**Theorem 2.3** ([6]) *Let  $X \in \underline{\mathcal{U}}$  be bounded,  $\mathbf{H} \in \mathcal{H}$ , and set  $\widehat{X}_t := X_t \mathbf{1}_{\{t < \mathbf{H}\}} + X_{\mathbf{H}-} \mathbf{1}_{\{t \geq \mathbf{H}\}}$ . Define*

$$Y := \overline{\mathcal{S}}^L[\widehat{X}] \quad \text{and} \quad \tau^* := \inf\{t \geq 0 : Y_t = \widehat{X}_t\}.$$

*Then  $Y_{\tau^*} = \widehat{X}_{\tau^*}$ ,  $Y$  is an  $\overline{\mathcal{E}}^L$ -supermartingale on  $[0, \mathbf{H}]$ , and an  $\overline{\mathcal{E}}^L$ -martingale on  $[0, \tau^*]$ . Consequently,  $\tau^*$  is an optimal stopping time.*

## 2.3 The derivatives

For  $s \in [0, T)$ ,  $u \in C^0(\Lambda^s)$ , we define its right time-derivative, if it exists, as in Dupire [4]:

$$\partial_t u(t, \omega) := \lim_{h \downarrow 0} \frac{u(t+h, \omega_{\cdot \wedge t}) - u(t, \omega)}{h}, \quad t \in [s, T) \quad \text{and} \quad \partial_t u(T, \omega) := \lim_{t \uparrow T} \partial_t u(t, \omega). \quad (2.8)$$

We next recall from [7] our definition of smooth processes, and we notice the difference with [5]. For  $\mathbf{H} \in \mathcal{H}^t$ , denote

$$\Lambda^t(\mathbf{H}) := \{(s, \omega) \in \Lambda : t \leq s < \mathbf{H}(\omega)\} \quad \text{and} \quad \bar{\Lambda}^t(\mathbf{H}) := \{(s, \omega) \in \Lambda : t \leq s \leq \mathbf{H}(\omega)\},$$

and we drop the  $t$  superscript whenever  $t = 0$ . We define the spaces  $C^0(\Lambda^t(\mathbf{H}))$  and  $C^0(\bar{\Lambda}^t(\mathbf{H}))$  in a natural way. Moreover, since  $\Lambda^t(\mathbf{H})$  is an open subset of  $(\Lambda^t, \mathbf{d}_\infty)$  and, for  $u : \bar{\Lambda}^t(\mathbf{H}) \rightarrow \mathbb{R}$ ,  $(t, \omega) \in \Lambda^t(\mathbf{H})$ , the time derivative  $\partial_t u(t, \omega)$  is well defined by (2.8), whenever the limit exists.

**Definition 2.4** *We say  $u \in C^{1,2}(\bar{\Lambda}^t(\mathbf{H}))$  if  $u \in C^0(\bar{\Lambda}^t(\mathbf{H}))$ ,  $\partial_t u \in C^0(\Lambda^t(\mathbf{H}))$ , and there exist  $\partial_\omega u \in C^0(\Lambda^t(\mathbf{H}), \mathbb{R}^d)$ ,  $\partial_{\omega\omega}^2 u \in C^0(\Lambda^t(\mathbf{H}), \mathbb{S}^d)$  such that, for any  $(s, \omega) \in [t, T) \times \Omega^t$ ,  $u^{s,\omega}$  has the following decomposition on  $\Lambda^t(\mathbf{H})$ :*

$$du^{s,\omega} = (\partial_t u)^{s,\omega} dt + (\partial_\omega u)^{s,\omega} \cdot dB^s + \frac{1}{2} (\partial_{\omega\omega}^2 u)^{s,\omega} : d\langle B^s \rangle, \quad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}_\infty^s. \quad (2.9)$$

We define  $C^{1,2}(\Lambda^t) := C^{1,2}(\bar{\Lambda}^t(T))$ . It is clear that, for any  $(t, \omega)$  and  $u \in C^{1,2}(\Lambda)$ , we have  $u^{t,\omega} \in C^{1,2}(\Lambda^t)$ .

By a direct localization argument, we see that the above  $\partial_\omega u$  and  $\partial_{\omega\omega}^2 u$ , if they exist, are unique. Consequently, we call them the first order and second order space derivatives of  $u$ , respectively.

For technical reasons, we shall extend the space  $C^{1,2}(\Lambda)$  slightly as follows. Let  $O \subset \mathbb{R}^d$  be an open and convex set containing  $\mathbf{0}$  and define  $H_i := H_i^{t,O}$  by

$$H_0 := t, \quad H_{i+1} := \inf\{s \geq H_i : B_s - B_{H_i} \notin O\} \wedge T, \quad i \geq 0, \quad (2.10)$$

One can easily check that

**Lemma 2.5** *For each  $i$ ,  $H_{i+1}^{H_i(\omega),\omega} \in \mathcal{H}^{H_i(\omega)}$ . Moreover, the set  $\{i : H_i(\omega) < T\}$  is finite for each  $\omega$ , and  $\lim_{i \rightarrow \infty} C_t^L[H_i < T] = 0$  for any  $L > 0$ .*

**Definition 2.6** *Let  $t \in [0, T]$ ,  $u : \Lambda^t \rightarrow \mathbb{R}$ . We say  $u \in \bar{C}^{1,2}(\Lambda^t)$  if there exists an open and convex set  $O \subset \mathbb{R}^d$  containing  $\mathbf{0}$  such that, for the  $H_i$  in (2.10) and for each  $i$ ,*

- (i)  $u^{H_i(\omega),\omega} \in C^{1,2}(\bar{\Lambda}^{H_i(\omega)}(H_{i+1}^{H_i(\omega),\omega}))$ , and  $u, \partial_t u, \partial_\omega u, \partial_{\omega\omega}^2 u$  are all bounded on  $[0, H_i]$ ;
- (ii) *there exist a partition  $\{E_j^i, j \geq 1\} \subset \mathcal{F}_{H_i}$  of  $\Omega$  and  $\varphi_j^i, \psi_j^i \in \text{UC}(\Lambda)$  with a modulus of continuity function  $\rho^i$ , which may depend on  $i$  but does not depend on  $j$ , such that*

$$u(H_{i+1}, \omega) = \sum_{j \geq 1} \left[ \varphi_j^i(H_i, B) + \psi_j^i(H_{i+1} - H_i, B_{H_{i+1}} - B_{H_i}) \right] \mathbf{1}_{E_j^i}. \quad (2.11)$$

We note that the space  $\bar{C}^{1,2}(\Lambda^t)$  here is slightly different from that in [5] and that in [7]. In particular, the technical requirement (2.11) is mainly for the partial comparison principle Proposition 4.1 below. For  $u \in \bar{C}^{1,2}(\Lambda^t)$  and  $\mathbb{P} \in \mathcal{P}_\infty^t$ , it is clear that the process  $u$  is a local  $\mathbb{P}$ -semimartingale on  $[t, T]$ , a  $\mathbb{P}$ -semimartingale on  $[0, H_i]$  for all  $i$ , and

$$du_s = \partial_t u_s ds + \frac{1}{2} \partial_{\omega\omega}^2 u_s : d\langle B^t \rangle_s + \partial_\omega u_s \cdot dB_s^t, \quad t \leq s < T, \quad \mathbb{P}\text{-a.s.}$$

## 2.4 Path dependent fully nonlinear PDEs

Following the accompanying paper [7], we continue our study of the following fully nonlinear parabolic path-dependent partial differential equation (PPDE, for short):

$$\mathcal{L}u(t, \omega) := \{-\partial_t u - G(\cdot, u, \partial_\omega u, \partial_{\omega\omega}^2 u)\}(t, \omega) = 0, \quad (t, \omega) \in \Lambda, \quad (2.12)$$

where the generator  $G : \Lambda \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \rightarrow \mathbb{R}$  satisfies the same conditions as in [7]:

**Assumption 2.7** *The nonlinearity  $G$  is nondecreasing in  $\gamma$  and satisfies:*

- (i) *For fixed  $(y, z, \gamma)$ ,  $G(\cdot, y, z, \gamma)$  is  $\mathbb{F}$ -progressively measurable, and  $|G(\cdot, 0, \mathbf{0}, \mathbf{0})| \leq C_0$ .*
- (ii)  *$G$  is continuous in  $(t, \omega)$  under  $d_\infty$ .*
- (iii)  *$G$  is uniformly Lipschitz continuous in  $(y, z, \gamma)$ , with a Lipschitz constant  $L_0$ .*

However, for our main wellposedness result, we shall also use the following stronger regularity requirement:

**Assumption 2.8**  *$G$  is uniformly continuous in  $(t, \omega)$  under  $d_\infty$  with a modulus of continuity function  $\rho_0$ .*

This condition is needed for our uniform approximation of  $G$  at below. We should point out though, for the semi-linear PPDE and path dependent HJB considered in [7] Section 4, this condition is violated when  $\sigma$  depends on  $\omega$ . We shall address this issue in some future research.

For all  $L > 0$ ,  $(t, \omega) \in \Lambda$  with  $t < T$ , and bounded  $\mathbb{F}$ -progressively measurable process  $u$ , define

$$\begin{aligned} \underline{\mathcal{A}}^L u(t, \omega) &:= \left\{ \varphi \in C_b^{1,2}(\Lambda^t) : (\varphi - u^{t,\omega})_t(\mathbf{0}) = \underline{\mathcal{S}}_t^L[(\varphi - u^{t,\omega})_{\cdot \wedge H}] \text{ for some } H \in \mathcal{H}^t \right\}, \\ \overline{\mathcal{A}}^L u(t, \omega) &:= \left\{ \varphi \in C_b^{1,2}(\Lambda^t) : (\varphi - u^{t,\omega})_t(\mathbf{0}) = \overline{\mathcal{S}}_t^L[(\varphi - u^{t,\omega})_{\cdot \wedge H}] \text{ for some } H \in \mathcal{H}^t \right\}, \end{aligned} \quad (2.13)$$

where  $\overline{\mathcal{S}}^L$  and  $\underline{\mathcal{S}}^L$  are the nonlinear Snell envelopes defined in (2.7).

**Definition 2.9** (i) *Let  $L > 0$ . We say  $u \in \underline{\mathcal{U}}$  (resp.  $\overline{\mathcal{U}}$ ) is a viscosity  $L$ -subsolution (resp.  $L$ -supersolution) of PPDE (2.12) if, for any  $(t, \omega) \in [0, T) \times \Omega$  and any  $\varphi \in \underline{\mathcal{A}}^L u(t, \omega)$  (resp.  $\varphi \in \overline{\mathcal{A}}^L u(t, \omega)$ ):*

$$\{ -\partial_t \varphi - G^{t,\omega}(\cdot, u, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) \}(t, \mathbf{0}) \leq \quad (\text{resp. } \geq) \quad 0.$$

- (ii) *We say  $u \in \underline{\mathcal{U}}$  (resp.  $\overline{\mathcal{U}}$ ) is a viscosity subsolution (resp. supersolution) of PPDE (2.12) if  $u$  is viscosity  $L$ -subsolution (resp.  $L$ -supersolution) of PPDE (2.12) for some  $L > 0$ .*
- (iii) *We say  $u \in \text{UC}_b(\Lambda)$  is a viscosity solution of PPDE (2.12) if it is both a viscosity subsolution and a viscosity supersolution.*

### 3 Main results

#### 3.1 Path-frozen PDE

For any  $(t, \omega) \in \Lambda$ , define the following deterministic function on  $[t, \infty) \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ :

$$g^{t,\omega}(s, y, z, \gamma) := G(s \wedge T, \omega_{\cdot \wedge t}, y, z, \gamma).$$



For any  $\varepsilon > 0$  and  $\eta \geq 0$ , we denote  $T_\eta := (1 + \eta)T$ , and

$$\begin{aligned} O_\varepsilon &:= \{x \in \mathbb{R}^d : |x| < \varepsilon\}, \quad \overline{O}_\varepsilon := \{x \in \mathbb{R}^d : |x| \leq \varepsilon\}, \quad \partial O_\varepsilon := \{x \in \mathbb{R}^d : |x| = \varepsilon\}; \\ \mathcal{O}_t^{\varepsilon, \eta} &:= [t, T_\eta] \times O_\varepsilon, \quad \overline{\mathcal{O}}_t^{\varepsilon, \eta} := [t, T_\eta] \times \overline{O}_\varepsilon, \quad \partial \mathcal{O}_t^{\varepsilon, \eta} := ([t, T_\eta] \times \partial O_\varepsilon) \cup (\{T_\eta\} \times O_\varepsilon), \end{aligned} \quad (3.1)$$

and we further simplify the notation for  $\eta = 0$ :

$$\mathcal{O}_t^\varepsilon := \mathcal{O}_t^{\varepsilon, 0}, \quad \overline{\mathcal{O}}_t^\varepsilon := \overline{\mathcal{O}}_t^{\varepsilon, 0}, \quad \partial \mathcal{O}_t^\varepsilon := \partial \mathcal{O}_t^{\varepsilon, 0}.$$

Our additional assumption is formulated on the following localized and path-frozen PDE defined for every  $(t, \omega) \in \Lambda$ :

$$(E)_{\varepsilon, \eta}^{t, \omega} \quad \mathbf{L}^{t, \omega} v := -\partial_t v - g^{t, \omega}(s, v, Dv, D^2 v) = 0 \quad \text{on } \mathcal{O}_t^{\varepsilon, \eta}. \quad (3.2)$$

Notice that for fixed  $(t, \omega)$ , this is a standard deterministic partial differential equation for which we now assume some wellposedness conditions.

**Assumption 3.1** *For any  $\varepsilon > 0, \eta \geq 0$ ,  $(t, \omega) \in \Lambda$ , and any  $h \in C^0(\partial \mathcal{O}_t^{\varepsilon, \eta})$ , we have  $\overline{v} = \underline{v}$ , where*

$$\begin{aligned} \overline{v}(s, x) &:= \inf \left\{ w(s, x) : w \text{ classical supersolution of } (E)_{\varepsilon, \eta}^{t, \omega} \text{ and } w \geq h \text{ on } \partial \mathcal{O}_t^{\varepsilon, \eta} \right\}, \\ \underline{v}(s, x) &:= \sup \left\{ w(s, x) : w \text{ classical subsolution of } (E)_{\varepsilon, \eta}^{t, \omega} \text{ and } w \leq h \text{ on } \partial \mathcal{O}_t^{\varepsilon, \eta} \right\}. \end{aligned} \quad (3.3)$$

We first note that the above sets of  $w$  are not empty. Indeed, one can check straightforwardly that, for any  $\delta > 0$  and denoting  $\lambda_\delta := \frac{C_0 + L_0 \|h\|_\infty}{\delta} + L_0$ ,

$$\overline{w}(t, x) := \|h\|_\infty + \delta e^{\lambda_\delta(T-t)}, \quad \underline{w}(t, x) := -\|h\|_\infty - \delta e^{\lambda_\delta(T-t)}, \quad (3.4)$$

satisfy the requirement for  $\overline{v}(s, x)$  and  $\underline{v}(s, x)$ , respectively. We also observe that our definition of  $\overline{v}$  and  $\underline{v}$  (3.3) is different from the corresponding definition in the standard Perron's approach, in which the  $w$  is a viscosity supersolution or subsolution. Loosely speaking, assuming that the comparison result holds for the equation  $(E)_{\varepsilon, \eta}^{t, \omega}$ , our Assumption 3.1 (ii) requires that the viscosity solution of  $(E)_{\varepsilon, \eta}^{t, \omega}$  can be approximated by a sequence of classical supersolutions and a sequence of classical subsolutions. We shall discuss further this issue in Section 7 below. In particular, we will provide some sufficient conditions for Assumption 3.1 to hold.

For later use, we show that Assumption 3.1 implies the wellposedness of  $(E)_{\varepsilon, \eta}^{t, \omega}$ .

**Proposition 3.2** *Let Assumptions 2.7 and 3.1 hold. Then, for all  $\varepsilon > 0, \eta \geq 0$ ,  $(t, \omega) \in \Lambda$ :*

(i) *Let  $v^1, v^2 \in C^0(\overline{\mathcal{O}}_t^{\varepsilon, \eta})$  be viscosity subsolution and supersolution, respectively, of the PDE*

- (E) $_{\varepsilon,\eta}^{t,\omega}$  with  $v^1 \leq v^2$  on  $\partial\mathcal{O}_t^{\varepsilon,\eta}$ . Then  $v^1 \leq v^2$  in  $\overline{\mathcal{O}_t^{\varepsilon,\eta}}$ .
- (ii) The PDE (E) $_{\varepsilon,\eta}^{t,\omega}$ , with boundary condition defined by an arbitrary function  $h \in C^0(\partial\mathcal{O}_t^{\varepsilon,\eta})$ , has a unique viscosity solution  $v = \overline{v} = \underline{v}$ .

**Proof** (i) First, by the same argument as Step 1 of the proof of Proposition 5.1 in [7], we see that the partial comparison principle holds for (E) $_{\varepsilon,\eta}^{t,\omega}$  under Assumption 2.7. That is, if  $v^1$  or  $v^2$  is in  $C^{1,2}(\overline{\mathcal{O}_t^{\varepsilon,\eta}})$ , then  $v^1 \leq v^2$  in  $\overline{\mathcal{O}_t^{\varepsilon,\eta}}$ . Now for general viscosity subsolution  $v^1$  and viscosity supersolution  $v^2$ , by partial comparison principle we have  $v^1 \leq \overline{v}$  and  $\underline{v} \leq v^2$ . Then it follows from Assumption 3.1 that  $v^1 \leq \overline{v} = \underline{v} \leq v^2$ .

(ii) By standard arguments, see e.g. Proposition 6.2 below for path dependent case, one can show that  $\overline{v}$  and  $\underline{v}$  are viscosity supersolution and subsolution of PDE (E) $_{\varepsilon,\eta}^{t,\omega}$  with boundary condition  $h$ . Then  $v := \overline{v} = \underline{v}$  is a viscosity solution, by Assumption 3.1. The uniqueness follows from the comparison principle.  $\blacksquare$

### 3.2 Wellposedness

**Assumption 3.3**  $\xi$  is uniformly continuous in  $\omega$  under  $\|\cdot\|_T$  with the modulus of continuity function  $\rho_0$ .

The main result of this paper is:

**Theorem 3.4** Let Assumptions 2.7, 2.8, 3.1, and 3.3 hold true.

- (i) Let  $u^1$  be a bounded viscosity subsolution and  $u^2$  a bounded viscosity supersolution of PPDE (2.12) with  $u^1(T, \cdot) \leq \xi \leq u^2(T, \cdot)$ . Then  $u^1 \leq u^2$  on  $\Lambda$ .
- (ii) The PPDE (2.12) with terminal condition  $\xi$  has a unique viscosity solution  $u \in UC_b(\Lambda)$ .

The proof is reported in Sections 5 and 6. A key ingredient is the partial comparison result, proved in Section 4, which extends the corresponding result in Theorem 5.3 of [7] to the set  $\overline{\mathcal{C}^{1,2}}(\Lambda)$ .

**Remark 3.5** In [7] Section 6, we transformed backward stochastic PDEs into a PPDE. It is straightforward to translate the assumptions in Theorem 3.4 to that setting and thus obtain the wellposedness of viscosity solutions of BSPDEs.  $\blacksquare$

### 3.3 A change variable formula

We have previously observed in [7], Proposition 3.14, that the classical change of variable formula is not known to hold true for our notion of viscosity solutions under Assumption

2.7. We now show that the additional Assumption 3.1 allows to prove that the change of variable holds true.

Let  $u \in C_b^{1,2}(\Lambda)$  and  $\Phi \in C^{1,2}([0, T] \times \mathbb{R})$ . Assume  $\Phi$  is strictly increasing in  $x$  and let  $\Psi$  denote its inverse function. Define

$$w(t, \omega) := \Phi(t, u(t, \omega)) \quad \text{and thus} \quad u(t, \omega) = \Psi(t, w(t, \omega)). \quad (3.5)$$

Then

$$\partial_t u = \Psi_t + \Psi_x \partial_t w, \quad \partial_\omega u = \Psi_x \partial_\omega w, \quad \partial_{\omega\omega}^2 u = \Psi_{xx} (\partial_\omega w)^2 + \Psi_x \partial_{\omega\omega}^2 w.$$

One can check straightforwardly that

$$\mathcal{L}u(t, \omega) = \Psi_x(t, w(t, \omega)) \tilde{\mathcal{L}}w(t, \omega) \quad \text{and} \quad \tilde{\mathcal{L}}w := -\partial_t w - \tilde{G}(t, \omega, w, \partial_\omega w, \partial_{\omega\omega}^2 w), \quad (3.6)$$

where

$$\tilde{G}(t, \omega, y, z, \gamma) := \frac{\Psi_t(t, y) + G(t, \omega, \Psi(t, y), \Psi_x(t, y)z, \Psi_{xx}(t, y)z^2 + \Psi_x(t, y)\gamma)}{\Psi_x(t, y)}.$$

Note that  $\Psi$  is increasing in  $x$  and  $\Psi_x > 0$ . Then

**Proposition 3.6** *Under the above assumptions on  $\Psi$ :*

- (i)  $u$  is classical solution (resp. supersolution, subsolution) of  $\mathcal{L}u = 0$  if and only if  $w := \Phi(t, u)$  is a classical solution (resp. supersolution, subsolution) of  $\tilde{\mathcal{L}}w = 0$ .
- (ii)  $\overline{w} = \Phi(t, \overline{u})$ ,  $\underline{w} = \Phi(t, \underline{u})$ , where

$$\begin{aligned} \overline{w}(t, \omega) &:= \inf \left\{ \psi(t, \mathbf{0}) : \psi \in \overline{C}^{1,2}(\Lambda^t), \tilde{\mathcal{L}}\psi \geq 0, \psi(T, \cdot) \geq \Phi(T, \xi^{t, \omega}) \right\}; \\ \underline{w}(t, \omega) &:= \sup \left\{ \psi(t, \mathbf{0}) : \psi \in \overline{C}^{1,2}(\Lambda^t), \tilde{\mathcal{L}}\psi \leq 0, \psi(T, \cdot) \leq \Phi(T, \xi^{t, \omega}) \right\}. \end{aligned}$$

**Theorem 3.7** *Assume both  $(G, \xi)$  and  $(\tilde{G}, \Phi(T, \xi))$  satisfy Assumptions 2.7, 2.8, 3.1, and 3.3. Then  $u$  is the viscosity solution of PPDE (2.12) with terminal condition  $\xi$  if and only if  $w := \Phi(t, u)$  is the viscosity solution of PPDE (3.6) with terminal condition  $\tilde{\xi} := \Phi(T, \xi)$ .*

We shall remark though that the above operator  $\tilde{G}$  is quadratic in the  $z$ -variable, so we need somewhat stronger conditions to ensure the wellposedness.

## 4 Partial comparison of viscosity solutions

In this section, we prove a partial comparison principle, i.e. a comparison result of a viscosity super- (resp. sub-) solution and a classical sub- (resp. super-) solution. This result extends Proposition 5.3 of [7] and is a first key step for our comparison principle. The proof is crucially based on the optimal stopping problem reported in Theorem 2.3.

**Proposition 4.1** *Assume Assumption 2.7. Let  $u^1 \in \underline{\mathcal{U}}$  be a viscosity subsolution and  $u^2 \in \overline{\mathcal{U}}$  a viscosity supersolution of PPDE (2.12). If  $u^1(T, \cdot) \leq u^2(T, \cdot)$  and one of  $u^1$  and  $u^2$  is in  $\bar{C}^{1,2}(\Lambda)$ , then  $u^1 \leq u^2$  on  $\Lambda$ .*

**Proof** We shall only prove  $u_0^1 \leq u_0^2$ . The inequality for general  $t$  can be proved similarly. Without loss of generality, we assume  $u^1$  and  $u^2$  are viscosity  $L$ -subsolution and  $L$ -supersolution, respectively, and  $u^1 \in \bar{C}^{1,2}(\Lambda)$  with corresponding hitting times  $H_i$ ,  $i \geq 0$ . By Proposition 3.6 (or Proposition 3.14 of [7]), we may assume without loss of generality that

$$G(t, \omega, y_1, z, \gamma) - G(t, \omega, y_2, z, \gamma) \geq y_2 - y_1 \quad \text{for all } y_1 \leq y_2. \quad (4.1)$$

We now prove the proposition in three steps.

*Step 1.* We first show that, for all  $i \geq 0$  and  $\omega \in \Omega$ ,

$$(u^1 - u^2)_{H_i}^+(\omega) \leq \bar{\mathcal{E}}_{H_i(\omega)}^L \left[ ((u^1)_{H_{i+1}}^{H_i, \omega} - (u^2)_{H_{i+1}-}^{H_i, \omega})^+ \right]. \quad (4.2)$$

Clearly it suffices to consider  $i = 0$ . Assume to the contrary that

$$2Tc := (u^1 - u^2)_0^+(\mathbf{0}) - \bar{\mathcal{E}}_0^L[(u_{H_1}^1 - u_{H_1-}^2)^+] > 0.$$

Define

$$\begin{aligned} X_t &:= (u^1 - u^2)_t^+ + ct, \quad \hat{X}_t := X_t \mathbf{1}_{\{t < H_1\}} + X_{H_1} \mathbf{1}_{\{t \geq H_1\}}, \quad t \in [0, T]; \\ Y &:= \bar{\mathcal{S}}^L[\hat{X}], \quad \tau^* := \inf\{t \geq 0 : Y_t = \hat{X}_t\}. \end{aligned}$$

Since  $u^1$  is bounded from above and  $u^2$  is bounded from below, by the definition of  $\underline{\mathcal{U}}$  and  $\overline{\mathcal{U}}$ , it follows that  $X$  is a bounded process in  $\underline{\mathcal{U}}$ . Applying Theorem 2.3 we have

$$\bar{\mathcal{E}}_0^L[\hat{X}_{\tau^*}] = Y_0 \geq X_0 = (u^1 - u^2)_0^+(\mathbf{0}) = 2Tc + \bar{\mathcal{E}}_0^L[(u_{H_1}^1 - u_{H_1-}^2)^+] \geq Tc + \bar{\mathcal{E}}_0^L[\hat{X}_{H_1}].$$

Then there exists  $\omega^* \in \Omega$  such that  $t^* := \tau^*(\omega^*) < H_1(\omega^*)$ . Next, by the  $\bar{\mathcal{E}}^L$ -supermartingale property of  $Y$  of Theorem 2.3, we have:

$$(u^1 - u^2)^+(t^*, \omega^*) + ct^* = X_{t^*}(\omega^*) = Y_{t^*}(\omega^*) \geq \bar{\mathcal{E}}_{t^*}^L[X_{H_1}^{t^*, \omega^*}] \geq \bar{\mathcal{E}}_{t^*}^L[cH_1^{t^*, \omega^*}] > ct^*,$$

implying that  $0 < (u^1 - u^2)^+(t^*, \omega^*) = (u^1 - u^2)(t^*, \omega^*)$ . Since  $u^1 - u^2 \in \underline{\mathcal{U}}$ , there exists  $H \in \mathcal{H}^{t^*}$  such that

$$H < H_1^{t^*, \omega^*} \quad \text{and} \quad (u^1 - u^2)_t^{t^*, \omega^*} > 0 \quad \text{for all } t \in [t^*, H]. \quad (4.3)$$

Then  $X_t^{t^*, \omega^*} = \varphi_t - (u^2)_t^{t^*, \omega^*}$  for all  $t \in [t^*, H]$ , where  $\varphi(t, \omega) := (u^1)^{t^*, \omega^*}(t, \omega) + ct$ . Observe that  $\varphi \in C^{1,2}(\Lambda^{t^*}(\mathbb{H}_1^{t^*, \omega^*}))$ , a consequence of our assumption  $u^1 \in C^{1,2}(\Lambda(\mathbb{H}_1))$ . Using again  $\bar{\mathcal{E}}^L$ -supermartingale property of  $Y$  of Theorem 2.3, we see that for all  $\tau \in \mathcal{T}^{t^*}$ :

$$(\varphi - (u^2)^{t^*, \omega^*})_{t^*} = X_{t^*}(\omega^*) = Y_{t^*}(\omega^*) \geq \bar{\mathcal{E}}_{t^*}^L[Y_{\tau \wedge H}^{t^*, \omega^*}] \geq \bar{\mathcal{E}}_{t^*}^L[X_{\tau \wedge H}^{t^*, \omega^*}] = \bar{\mathcal{E}}_{t^*}^L[(\varphi - (u^2)^{t^*, \omega^*})_{\tau \wedge H}].$$

That is,  $\varphi \in \bar{\mathcal{A}}^L u^2(t^*, \omega^*)$ , and by the viscosity  $L$ -supersolution property of  $u^2$ :

$$\begin{aligned} 0 \leq \{ -\partial_t \varphi - G(\cdot, u^2, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) \}(t^*, \omega^*) &= -c - \{ \partial_t u^1 + G(\cdot, u^2, \partial_\omega u^1, \partial_{\omega\omega}^2 u^1) \}(t^*, \omega^*) \\ &\leq -c - \{ \partial_t u^1 + G(\cdot, u^1, \partial_\omega u^1, \partial_{\omega\omega}^2 u^1) \}(t^*, \omega^*), \end{aligned}$$

where the last inequality follows from (4.3) and (4.1). Since  $c > 0$ , this is in contradiction with the subsolution property of  $u^1$ , and thus we prove (4.2).

**Remark 4.2** *The rest of the proof is only needed in the case where  $u^1 \in \bar{C}^{1,2}(\Lambda) \setminus C^{1,2}(\Lambda)$ . Indeed, if  $u^1 \in C^{1,2}(\Lambda)$ , then  $H_1 = T$  and it follows from Step 1 that  $(u^1 - u^2)_0^+ \leq \bar{\mathcal{E}}_0^L[(u_T^1 - u_{T-}^2)^+] \leq \bar{\mathcal{E}}_0^L[(u_T^1 - u_T^2)^+] = 0$ , and then  $u_0^1 \leq u_0^2$ . In fact, this is the partial comparison principle proved in [7] Proposition 5.3.*

*Step 2.* We shall prove below that, for any  $i \geq 1$  and any  $\mathbb{P} \in \mathcal{P}_L$ ,

$$\Delta_i := (u^1 - u^2)_{\mathbb{H}_i-}^+ - \text{ess-sup}_{\mathbb{P}' \in \mathcal{P}_L(\mathbb{P}, \mathbb{H}_i)} \mathbb{E}^{\mathbb{P}'}[(u^1 - u^2)_{\mathbb{H}_{i+1}-}^+ | \mathcal{F}_{\mathbb{H}_i}] \leq 0, \quad \mathbb{P}\text{-a.s.} \quad (4.4)$$

where  $\mathcal{P}_L(\mathbb{P}, \mathbb{H}_i) := \{\mathbb{P}' \in \mathcal{P}_L : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_{\mathbb{H}_i}\}$ . Then by standard arguments, we have

$$\mathbb{E}^{\mathbb{P}}[(u^1 - u^2)_{\mathbb{H}_i-}^+] \leq \sup_{\mathbb{P}' \in \mathcal{P}_L(\mathbb{P}, \mathbb{H}_i)} \mathbb{E}^{\mathbb{P}'}[(u^1 - u^2)_{\mathbb{H}_{i+1}-}^+] \leq \bar{\mathcal{E}}_0^L[(u^1 - u^2)_{\mathbb{H}_{i+1}-}^+].$$

Since  $\mathbb{P} \in \mathcal{P}_L$  is arbitrary, this leads to  $\bar{\mathcal{E}}_0^L[(u^1 - u^2)_{\mathbb{H}_i-}^+] \leq \bar{\mathcal{E}}_0^L[(u^1 - u^2)_{\mathbb{H}_{i+1}-}^+]$ , and by induction,  $(u^1 - u^2)_0^+ \leq \bar{\mathcal{E}}_0^L[(u^1 - u^2)_{\mathbb{H}_i-}^+]$ , for all  $i$ . Notice that  $(u^1 - u^2)^+$  is bounded,  $\mathcal{C}_0^L[\mathbb{H}_i < T] \rightarrow 0$  as  $i \nearrow \infty$ , by Lemma 2.5, and  $u_{T-}^2 \geq u_T^2$  by the definition of  $\bar{U}$ . Then, sending  $i \rightarrow \infty$ , we obtain  $(u^1 - u^2)_0^+ \leq \bar{\mathcal{E}}_0^L[(u^1 - u^2)_{T-}^+] \leq \bar{\mathcal{E}}_0^L[(u^1 - u^2)_T^+] = 0$ , which completes the proof of  $u_0^1 \leq u_0^2$ .

*Step 3.* It remains to prove (4.4). We start with constructing some partitions of  $\Omega$ . Recall the partition  $\{E_j^i, j \geq 1\} \subset \mathcal{F}_{\mathbb{H}_i}$  and the uniform continuous mappings  $\varphi_j^i$  and  $\psi_j^i$  in (2.11) corresponding to  $u^1 \in \bar{C}^{1,2}(\Lambda)$ . Fix an arbitrarily small  $\delta > 0$ , let  $0 = t_0 < t_1 < \dots < t_n = T$  such that  $t_{k+1} - t_k \leq \delta$  for  $k = 0, \dots, n-1$ . Denote  $E_{k,j}^i := E_j^i \cap \{t_k \leq \mathbb{H}_i < t_{k+1}\}$ . Then  $\{E_{k,j}^i : k, j \geq 1\} \subset \mathcal{F}_{\mathbb{H}_i}$  form a new partition of  $\Omega$ . Since  $\Omega$  is separable, there exists a finer partition  $\{E_{k,j,l}^i : k, j, l \geq 1\} \subset \mathcal{F}_{\mathbb{H}_i}$  such that, for any  $\omega, \omega' \in E_{k,j,l}^i$ ,  $\|\omega_{\cdot \wedge \mathbb{H}_i(\omega)} - \omega'_{\cdot \wedge \mathbb{H}_i(\omega')}\| \leq \delta$ .

Similarly, we may form a partition  $\{E_{k,j,l}^{i-1} : k, j, l \geq 1\} \subset \mathcal{F}_{H_{i-1}}$ . Consider the finer partition  $E_{k,j,l}^i \cap E_{k',j',l'}^{i-1} \in \mathcal{F}_{H_i}$  over all possible  $(k, j, l)$  and  $(k', j', l')$ , and renumerate them as  $\tilde{E}_j^i$ .

Next, for each  $\tilde{E}_j^i$ , if there exists  $\omega^{i,j} \in \tilde{E}_j^i$  such that  $H_i(\omega^{i,j}) = \min_{\omega \in \tilde{E}_j^i} H_i(\omega)$ , we set  $\omega^{i,j,m} := \omega^{i,j}$  for all  $m \geq 1$ . Otherwise we find a sequence  $\omega^{i,j,m} \in \tilde{E}_j^i$  such that  $t_{j,m}^i := H_i(\omega^{i,j,m}) \downarrow \inf_{\omega \in \tilde{E}_j^i} H_i(\omega)$ . Denote  $t_{j,0}^i := t_{k+1}$  for the corresponding  $k$  such that  $t_k \leq H_i < t_{k+1}$  on  $\tilde{E}_j^i$ . We will use the following final partition:  $\tilde{E}_{j,m}^i := \tilde{E}_j^i \cap \{t_{j,m+1}^i \leq H_i < t_{j,m}^i\} \in \mathcal{F}_{H_i}$ . By renumeration and by abusing the notation, we may still denote them as  $\tilde{E}_j^i \in \mathcal{F}_{H_i}$  with corresponding  $\omega^j \in \tilde{E}_j^i$  such that  $\tilde{t}_j := H_i(\omega^j) = \min_{\omega \in \tilde{E}_j^i} H_i(\omega)$ .

Now fix an arbitrary  $\mathbb{P} \in \mathcal{P}_L$  and  $\varepsilon > 0$ . Denote

$$\hat{u} := u^1 - u^2.$$

Since  $u^2 \in \overline{\mathcal{U}}$ , we have  $u_{H_i-}^2 \geq u_{H_i}^2$ . Then, for each  $j$ , it follows from (4.2) that:

$$\hat{u}_{H_i-}^+(\omega^j) \leq \hat{u}_{H_i}^+(\omega^j) \leq \mathbb{E}^{\mathbb{P}_j} \left[ [\hat{u}^+]_{\tilde{t}_j, \omega^j}^{t_j, \omega^j} \right] + \varepsilon \quad \text{for some } \mathbb{P}_j \in \mathcal{P}_L^{\tilde{t}_j}.$$

Define  $\hat{\mathbb{P}} \in \mathcal{P}_L(\mathbb{P}, H_i)$  such that, for  $\mathbb{P}$ -a.s.  $\omega \in \tilde{E}_j^i$ , the  $\hat{\mathbb{P}}^{H_i(\omega), \omega}$ -distribution of  $B^{H_i}$  is equal to the  $\mathbb{P}_j$ -distribution of  $B^{\tilde{t}_j}$ , where  $\hat{\mathbb{P}}^{H_i(\omega), \omega}$  denotes the r.c.p.d. Then,  $\mathbb{P}$ -a.s. on  $\tilde{E}_j^i$ ,

$$\begin{aligned} \mathbb{E}^{\hat{\mathbb{P}}} \left[ \hat{u}_{H_{i+1}-}^+ | \mathcal{F}_{H_i} \right] (\omega) &= \mathbb{E}^{\hat{\mathbb{P}}^{H_i(\omega), \omega}} \left[ \hat{u}^+ (H_{i+1}(\omega \otimes_{H_i(\omega)} B^{H_i(\omega)})-, \omega \otimes_{H_i(\omega)} B^{H_i(\omega)}) \right] \\ &= \mathbb{E}^{\mathbb{P}_j} \left[ \hat{u}^+ (H_{i+1}(\omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j})-, \omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j}) \right], \end{aligned}$$

where  $\tilde{B}_s^{\tilde{t}_j} := B_{s-H_i(\omega)+\tilde{t}_j}^{\tilde{t}_j}$ ,  $s \geq H_i(\omega)$ . Thus,  $\mathbb{P}$ -a.s.

$$\begin{aligned} \Delta_i(\omega) &\leq \hat{u}_{H_i-}^+(\omega) - \mathbb{E}^{\hat{\mathbb{P}}} \left[ \hat{u}_{H_{i+1}-}^+ | \mathcal{F}_{H_i} \right] (\omega) \\ &\leq \sum_{j \geq 1} \left[ \hat{u}_{H_i-}^+(\omega) - \hat{u}_{H_i-}^+(\omega^j) \right] \mathbf{1}_{\tilde{E}_j^i}(\omega) + \varepsilon \\ &\quad + \sum_{j \geq 1} \mathbb{E}^{\mathbb{P}_j} \left[ [\hat{u}^+]_{\tilde{t}_j, \omega^j}^{t_j, \omega^j} - \hat{u}^+ (H_{i+1}(\omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j})-, \omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j}) \right] \mathbf{1}_{\tilde{E}_j^i}(\omega) \\ &\leq \sum_{j \geq 1} \left[ \hat{u}_{H_i-}(\omega) - \hat{u}_{H_i-}(\omega^j) \right]^+ \mathbf{1}_{\tilde{E}_j^i}(\omega) + \varepsilon \\ &\quad + \sum_{j \geq 1} \mathbb{E}^{\mathbb{P}_j} \left[ [\hat{u}^+]_{\tilde{t}_j, \omega^j}^{t_j, \omega^j} - \hat{u}^+ (H_{i+1}(\omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j})-, \omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j}) \right]^+ \mathbf{1}_{\tilde{E}_j^i}(\omega). \end{aligned} \tag{4.5}$$

We now estimate the above error for fixed  $\omega \in \tilde{E}_j^i$ . Recall that each  $\tilde{E}_j^i$  is a subset of some  $E_{k,j,l}^i \cap E_{k',j',l'}^{i-1}$ , then by the construction we see that  $\mathbf{d}_\infty((H_{i-1}(\omega), \omega), (H_{i-1}(\omega^j), \omega^j)) \leq 2\delta$ ,  $\mathbf{d}_\infty((H_i(\omega), \omega), (H_i(\omega^j), \omega^j)) \leq 2\delta$ . This implies further that

$$\begin{aligned} &\mathbf{d}_\infty \left( (H_i(\omega) - H_{i-1}(\omega), \omega_{H_{i-1}(\omega)+\cdot} - \omega_{H_{i-1}(\omega)}), (H_i(\omega^j) - H_{i-1}(\omega^j), \omega_{H_{i-1}(\omega^j)+\cdot}^j - \omega_{H_{i-1}(\omega^j)}^j) \right) \\ &\leq \mathbf{d}_\infty((H_{i-1}(\omega), \omega), (H_{i-1}(\omega^j), \omega^j)) + \mathbf{d}_\infty((H_i(\omega), \omega), (H_i(\omega^j), \omega^j)) + \eta_\delta \leq 4\delta + \eta_\delta(\omega), \end{aligned}$$

where

$$\eta_\delta(\omega) := \sup_{0 \leq t \leq T} \sup_{t \leq s \leq t+\delta} |\omega_s - \omega_t|. \quad (4.6)$$

Then, since  $u^1$  is continuous, by (2.11) we have

$$u_{H_i-}^1(\omega) - u_{H_i-}^1(\omega^j) = u_{H_i}^1(\omega) - u_{H_i}^1(\omega^j) \leq \rho^{i-1}(2\delta) + \rho^{i-1}(4\delta + \eta_\delta(\omega)). \quad (4.7)$$

Note that (2.10) leads to

$$H_{i+1}(\omega^j \otimes_{\tilde{t}_j} B^{\tilde{t}_j}) - \tilde{t}_j = H_{i+1}(\omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j}) - H_i(\omega). \quad (4.8)$$

Then similarly we have

$$\begin{aligned} & (u_{H_{i+1}-}^1)^{\tilde{t}_j, \omega^j} - u^1(H_{i+1}(\omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j})-, \omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j}) \\ &= u^1(H_{i+1}(\omega^j \otimes_{\tilde{t}_j} B^{\tilde{t}_j}), \omega^j \otimes_{\tilde{t}_j} \tilde{B}^{\tilde{t}_j}) - u^1(H_{i+1}(\omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j}), \omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j}) \leq 2\rho^i(2\delta). \end{aligned} \quad (4.9)$$

Moreover, let  $\rho$  denote the modulus of continuity function of  $-u^2 \in \underline{\mathcal{U}}$  in (2.2). Then:

$$\begin{aligned} u_{H_i-}^2(\omega^j) - u_{H_i-}^2(\omega) &= u^2(\tilde{t}_j-, \omega^j) - u^2(\tilde{t}_j-, \omega) + u^2(\tilde{t}_j-, \omega) - u^2(H_i(\omega)-, \omega) \\ &\leq \rho(\delta) + \sup_{H_i(\omega)-\delta \leq t \leq H_i(\omega)} [u^2(t-, \omega) - u^2(H_i(\omega)-, \omega)], \end{aligned} \quad (4.10)$$

and, recalling (4.8),

$$\begin{aligned} & u^2(H_{i+1}(\omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j})-, \omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j}) - (u_{H_{i+1}-}^2)^{\tilde{t}_j, \omega^j} \\ &= (-u^2)(H_{i+1}(\omega^j \otimes_{\tilde{t}_j} B^{\tilde{t}_j})-, \omega^j \otimes_{\tilde{t}_j} \tilde{B}^{\tilde{t}_j}) - (-u^2)(H_{i+1}(\omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j})-, \omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j}) \\ &\leq \rho(\mathbf{d}_\infty((H_{i+1}(\omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j}), \omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j}), (H_{i+1}(\omega^j \otimes_{\tilde{t}_j} B^{\tilde{t}_j}), \omega^j \otimes_{\tilde{t}_j} \tilde{B}^{\tilde{t}_j}))) \\ &\leq \rho(\mathbf{d}_\infty((H_i(\omega), \omega), (H_i(\omega^j), \omega^j)) + \eta_\delta(B^{\tilde{t}_j})) \leq \rho(2\delta + \eta_\delta(B^{\tilde{t}_j})). \end{aligned} \quad (4.11)$$

By (4.9), (4.11), and noting that that  $\hat{u}$  is bounded, we obtain

$$[\hat{u}_{H_{i+1}-}^{\tilde{t}_j, \omega^j} - \hat{u}(H_{i+1}(\omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j})-, \omega \otimes_{H_i(\omega)} \tilde{B}^{\tilde{t}_j})]^+ \leq 2\rho^i(2\delta) + \rho(2\delta + \eta_\delta(B^{\tilde{t}_j})) \wedge C.$$

Plug this and (4.7), (4.10) into (4.5), we obtain By the above estimate, we can now conclude that

$$\begin{aligned} \Delta_i(\omega) &\leq \rho^{i-1}(2\delta) + \rho^{i-1}(4\delta + \eta_\delta(\omega)) + 2\rho^i(2\delta) + \sum_{j \geq 1} \mathbb{E}^{\mathbb{P}^j} [\rho(2\delta + \eta_\delta(B^{\tilde{t}_j})) \wedge C] \mathbf{1}_{\tilde{E}_j^i}(\omega) \\ &\quad + \rho(\delta) + \sup_{H_i(\omega)-\delta \leq t \leq H_i(\omega)} [u^2(t-, \omega) - u^2(H_i(\omega)-, \omega)] + \varepsilon \\ &\leq \rho^{i-1}(2\delta) + \rho^{i-1}(4\delta + \eta_\delta(\omega)) + 2\rho^i(2\delta) + \bar{\mathcal{E}}_0^L [\rho(2\delta + \eta_\delta(B)) \wedge C] \\ &\quad + \rho(\delta) + \sup_{H_i(\omega)-\delta \leq t \leq H_i(\omega)} [u^2(t-, \omega) - u^2(H_i(\omega)-, \omega)] + \varepsilon, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

One can easily check that  $\lim_{\delta \rightarrow 0} \bar{\mathcal{E}}_0^L \left[ \rho \left( 2\delta + \eta_\delta(B) \right) \wedge C \right] = 0$ . Then by sending  $\delta \rightarrow 0$  and  $\varepsilon \rightarrow 0$  in the above estimate, we obtain (4.4) immediately.  $\blacksquare$

## 5 Comparison and Uniqueness

We follow Perron's approach. In light of (3.3), we define

$$\bar{u}(t, \omega) := \inf \{ \psi(t, \mathbf{0}) : \psi \in \bar{\mathcal{D}}(t, \omega) \}, \quad \underline{u}(t, \omega) := \sup \{ \psi(t, \mathbf{0}) : \psi \in \underline{\mathcal{D}}(t, \omega) \}, \quad (5.1)$$

where,

$$\begin{aligned} \bar{\mathcal{D}}(t, \omega) &:= \left\{ \psi \in \bar{C}^{1,2}(\Lambda^t) : \|\psi^-\|_\infty < \infty, (\mathcal{L}\psi)^{t,\omega} \geq 0 \text{ on } [t, T) \times \Omega^t \text{ and } \psi_T \geq \xi^{t,\omega} \right\}, \\ \underline{\mathcal{D}}(t, \omega) &:= \left\{ \psi \in \bar{C}^{1,2}(\Lambda^t) : \|\psi^+\|_\infty < \infty, (\mathcal{L}\psi)^{t,\omega} \leq 0 \text{ on } [t, T) \times \Omega^t \text{ and } \psi_T \leq \xi^{t,\omega} \right\}. \end{aligned} \quad (5.2)$$

By the partial comparison result of Proposition 4.1, it is clear that

$$\underline{u} \leq \bar{u}. \quad (5.3)$$

A crucial step for our proof is to show that equality holds in the last inequality under our additional assumptions.

**Proposition 5.1** *Under Assumptions 2.7, 2.8, 3.1, and 3.3, we have  $\bar{u} = \underline{u}$ .*

Before proving this result, we show how it implies the comparison result for the PPDE.

**Proof of Theorem 3.4 (i) and uniqueness.** By Proposition 4.1, we have  $u^1 \leq \bar{u}$  and  $\underline{u} \leq u^2$ . Then Proposition 5.1 implies  $u^1 \leq u^2$  immediately, which implies further the uniqueness of viscosity solution.  $\blacksquare$

In the rest of this subsection, we prove Proposition 5.1. We shall divide the proof into several lemmas. By Proposition 3.6, we shall always assume without loss of generality that  $G$  is strictly decreasing in  $y$ , i.e. (4.1).

We start with some estimates for viscosity solutions of PDE (3.2).

**Lemma 5.2** *Assume Assumptions 2.7 and 3.1. Let  $h^i : \partial\mathcal{O}_t^\varepsilon \rightarrow \mathbb{R}$  be continuous,  $i = 1, 2$ , and  $v^i$  be the viscosity solution of the PDE  $(E)_{\varepsilon,0}^{t,\omega}$  with boundary condition  $h^i$ . Then, denoting  $\delta v := v^1 - v^2$ ,  $\delta h := h^1 - h^2$ ,*

$$\delta v(s, x) \leq \bar{\mathcal{E}}_s^{L_0} \left[ (\delta h)^+(\mathbb{H}, x + B_\mathbb{H}^s) \right], \quad \text{where } \mathbb{H} := T \wedge \inf \{ r \geq s : |x + B_r^s| = \varepsilon \}. \quad (5.4)$$



**Proof** By standard results, or see [7] Proposition 4.8 for path dependent case, the function  $w(s, x) := \bar{\mathcal{E}}_s^{L_0} \left[ (\delta h)^+(\mathbb{H}, x + B_{\mathbb{H}}^s) \right]$  is a viscosity solution of the nonlinear PDE:

$$-\partial_t w - L_0 |Dw| - \frac{1}{2} \sup_{0 \leq \sigma \leq \sqrt{2L_0} I_d} \{ \sigma^2 : D^2 w \} = 0 \quad \text{on } \mathcal{O}_t^\varepsilon, \quad \text{and } w = (\delta h)^+ \quad \text{on } \partial \mathcal{O}_t^\varepsilon.$$

Let  $K$  be a smooth nonnegative kernel with unit total mass. For all  $\eta > 0$ , we define the molification  $w^\eta := w * K^\eta$  of  $w$ . Then  $w^\eta$  is smooth, and it follows from a convexity argument in Krylov [11] that  $w^\eta$  is a classical supersolution of

$$\begin{aligned} -\partial_t w^\eta - L_0 |Dw^\eta| - \frac{1}{2} \sup_{0 \leq \sigma \leq \sqrt{2L_0} I_d} \{ \sigma^2 : D^2 w^\eta \} &\geq 0 \quad \text{on } \mathcal{O}_t^\varepsilon, \\ w^\eta &= (\delta h) * K^\eta \quad \text{on } \partial \mathcal{O}_t^\varepsilon. \end{aligned} \tag{5.5}$$

We claim that

$$\tilde{w}^\eta + v^2 \text{ is a supersolution of the PDE (E)}_{\varepsilon,0}^{t,\omega}, \text{ where } \tilde{w}^\eta := w^\eta + \|w^\eta - (\delta h)^+\|_{\mathbb{L}^\infty(\partial \mathcal{O}_t^\varepsilon)} \tag{5.6}$$

Then, noting that  $\tilde{w}^\eta + v^2 = w^\eta + h^2 + \|w^\eta - (\delta h)^+\|_{\mathbb{L}^\infty(\partial \mathcal{O}_t^\varepsilon)} \geq h^1 = v^1$  on  $\partial \mathcal{O}_t^\varepsilon$ , we deduce from the comparison principle Proposition 3.2 (i) that  $\tilde{w}^\eta + v^2 \geq v^1$  on  $\overline{\mathcal{O}_t^\varepsilon}$ . Sending  $\eta \searrow 0$ , this implies that  $\delta v \leq w$ , which is the required result.

It remain to prove that  $\tilde{w}^\eta + v^2$  is a supersolution of the PDE  $(E)_{\varepsilon,0}^{t,\omega}$ . Let  $(t_0, x_0) \in \mathcal{O}_t^\varepsilon$ ,  $\phi \in C^{1,2}(\mathcal{O}_t^\varepsilon)$  be such that  $0 = (\phi - \tilde{w}^\eta - v^2)(t_0, x_0) = \max(\phi - \tilde{w}^\eta - v^2)$ . Then, it follows from the viscosity supersolution property of  $v^2$  that  $\mathbf{L}^{t,\varepsilon}(\phi - \tilde{w}^\eta)(t_0, x_0) \geq 0$ . Hence, at the point  $(t_0, x_0)$ , by (4.1) and (5.5) we have

$$\begin{aligned} \mathbf{L}^{t,\varepsilon} \phi &\geq \mathbf{L}^{t,\varepsilon} \phi - \mathbf{L}^{t,\varepsilon}(\phi - \tilde{w}^\eta) \\ &= -\partial_t w^\eta - g^{t,\omega}(\cdot, \phi, D\phi, D^2 \phi) + g^{t,\omega}(\cdot, \phi - \tilde{w}^\eta, D(\phi - w^\eta), D^2(\phi - w^\eta)) \\ &\geq -\partial_t w^\eta - g^{t,\omega}(\cdot, \phi, D\phi, D^2 \phi) + g^{t,\omega}(\cdot, \phi, D(\phi - w^\eta), D^2(\phi - w^\eta)) \\ &\geq L_0 |Dw^\eta| + \frac{1}{2} \sup_{0 \leq \sigma \leq \sqrt{2L_0} I_d} \{ \sigma^2 : D^2 w^\eta \} - \alpha \cdot Dw^\eta - \gamma : D^2 w^\eta \geq 0, \end{aligned}$$

where  $|\alpha| \leq L_0$  and  $0 \leq \gamma \leq L_0 I_d$ , thanks to Assumption 2.7. This proves (5.6).  $\blacksquare$

We next introduce some notations, as in [7] Section 7. Let  $t_0 = 0$ ,  $x_0 = 0$ ,  $(t_i)_{i \geq 1}$  an increasing sequence in  $(0, T)$ , and  $(x_i)_{i \geq 1}$  a sequence in  $\mathbb{R}^d$ . For all  $n \geq 0$ , set  $\pi_n := (t_i, x_i)_{0 \leq i \leq n}$ ,  $\omega_{t_i}^{\pi_n} := \sum_{j=0}^i x_j$ ,  $i \leq n$ , and let  $\omega^{\pi_n} \in \Omega$  be the linear interpolation of  $\{(t_i, \omega_{t_i}^{\pi_n})_{0 \leq i \leq n}, (T, \omega_{t_n}^{\pi_n})\}$ . Moreover, for  $t \in (t_n, T]$  and  $x \in \mathbb{R}^d$ , let  $\pi_n^{t,x} := (\pi_n, (t, x))$ , and denote similarly  $\omega^{\pi_n^{t,x}} \in \Omega$  as the linear interpolation of  $\{(t_i, \omega_{t_i}^{\pi_n})_{0 \leq i \leq n}, (t, \omega_{t_n}^{\pi_n} + x), (T, \omega_{t_n}^{\pi_n} + x)\}$ .

**Lemma 5.3** *Assume Assumptions 2.7, 3.1, and 3.3. Let  $|x_i| = \varepsilon$ ,  $i \geq 1$ , for some  $\varepsilon > 0$ . There exists a sequence of continuous functions  $(\pi_n, t, x) \mapsto \theta_n^\varepsilon(\pi_n, t, x)$ , bounded uniformly in  $(\varepsilon, n)$ , and such that:*

$$\begin{aligned} \theta_n^\varepsilon(\pi_n; \cdot) & \text{ is a viscosity solution of } (E)_{\varepsilon,0}^{t_n, \omega^{\pi_n}}, \\ \theta_n^\varepsilon(\pi_n; T, x) & = \xi(\omega^{\pi_n^{T,x}}), \quad |x| \leq \varepsilon \\ \theta_n^\varepsilon(\pi_n; t, x) & = \theta_{n+1}^\varepsilon(\pi_n^{t,x}; t, 0), \quad t \in (t_n, T), |x| = \varepsilon. \end{aligned} \tag{5.7}$$

**Proof** We proceed in two steps.

*Step 1.* Denote

$$\begin{aligned} \bar{g}(y, z, \gamma) & := \sup_{|\alpha| \leq L_0, \beta \leq \sqrt{2L_0}I_d} [\alpha \cdot z + \frac{1}{2}\beta^2 : \gamma] + L_0|y| + C_0; \\ \underline{g}(y, z, \gamma) & := \inf_{|\alpha| \leq L_0, \beta \leq \sqrt{2L_0}I_d} [\alpha \cdot z + \frac{1}{2}\beta^2 : \gamma] - L_0|y| - C_0 \end{aligned}$$

It is clear that

$$\underline{g} \leq G \leq \bar{g}.$$

We first prove the lemma in the cases  $G = \bar{g}$  and  $G = \underline{g}$ . Indeed, as in [7] Section 7 for semilinear PPDEs, in these cases we may have explicit representation for the required functions.

For any  $N$ , denote

$$\bar{\theta}_{N,N}^\varepsilon(\pi_N, t_N, \mathbf{0}) := \xi(\omega^{\pi_N, (T, \mathbf{0})}),$$

and for  $n = N_1, \dots, 0$ ,  $\theta := \bar{\theta}_{N,n}^\varepsilon(\pi_n; \cdot)$  is the unique viscosity solution of the following PDE:

$$-\partial_t \theta - \bar{g}(\theta, D\theta, D^2\theta) = 0 \text{ in } \mathcal{O}_{t_n}^\varepsilon, \quad \theta(t, x) = \bar{\theta}_{N,n+1}^\varepsilon(\pi_n, (t, x); t, \mathbf{0}) \text{ on } \partial\mathcal{O}_{t_n}^\varepsilon. \tag{5.8}$$

Then clearly  $\bar{\theta}_{N,n}^\varepsilon(\pi_n; t, x)$  are uniformly bounded and continuous in all variables  $(\pi_n, t, x)$ .

Following standard arguments, we have the following representation:

$$\bar{\theta}_{N,n}^\varepsilon(\pi_n, t, x) := \sup_{b \in \mathcal{B}_{L_0}^t} \bar{\mathcal{E}}_t^{L_0} \left[ e^{\int_t^{H_{N-n}} b_r dr} \xi \left( \omega^{\Pi_{N-n}^\varepsilon(\pi_n, t, x), (T, \mathbf{0})} \right) + C_0 \int_t^{H_{N-n}} e^{\int_t^s b_r dr} ds \right],$$

where,  $\mathcal{B}_{L_0}^t$  denotes the collection of all  $\mathbb{F}^t$ -progressively measurable scalar processes bounded by  $L_0$ , and for any  $(t, x) \in \bar{\mathcal{O}}_{t_n}^\varepsilon$ :

$$\begin{aligned} H_1 & := \inf\{s \geq t : |x + B_s^t| = \varepsilon\} \wedge T, \quad H_{i+1} := \{s > H_i : |B_s^t - B_{H_i}^t| = \varepsilon\} \wedge T, \quad i \geq 1; \\ \Pi_i^\varepsilon(\pi_n, t, x) & := (\pi_n, (H_1, x + B_{H_1}^t), \dots, (H_i, B_{H_i}^t - B_{H_{i-1}}^t)). \end{aligned} \tag{5.9}$$

Define

$$\bar{\theta}_n^\varepsilon(\pi_n, t, x) := \sup_{b \in \mathcal{B}_{L_0}^t} \bar{\mathcal{E}}_t^{L_0} \left[ e^{\int_t^T b_r dr} \overline{\lim}_{i \rightarrow \infty} \xi \left( \omega^{\Pi_i^\varepsilon(\pi_n, t, x)} \right) + C_0 \int_t^T e^{\int_t^s b_r dr} ds \right].$$

Then, by Lemma 2.5,

$$|\bar{\theta}_n^\varepsilon(\pi_n, t, x) - \bar{\theta}_{N,n}^\varepsilon(\pi_n, t, x)| \leq CC_{t_n}^{L_0} (\mathbf{H}_{N-n} < T) \rightarrow 0 \text{ as } N \rightarrow \infty.$$

This implies that  $\bar{\theta}_{N,n}^\varepsilon(\pi_n; t, x)$  are uniformly bounded and continuous in all variables  $(\pi_n, t, x)$ . Moreover, by stability of viscosity solutions we see that

$$\begin{aligned} \bar{\theta}_n^\varepsilon(\pi_n; \cdot) \text{ is the viscosity solution of PDE (5.8) in } \mathcal{O}_{t_n}^\varepsilon \text{ with boundary condition:} \\ \bar{\theta}_n^\varepsilon(\pi_n; T, x) = \xi(\omega^{\pi_n^{T,x}}), \quad |x| \leq \varepsilon, \quad \bar{\theta}_n^\varepsilon(\pi_n; t, x) = \bar{\theta}_{n+1}^\varepsilon(\pi_n^{t,x}; t, 0), \quad t \in (t_n, T), |x| = \varepsilon. \end{aligned}$$

Similarly we may define the following  $\underline{\theta}_n^\varepsilon$  satisfying the corresponding properties related  $\underline{G}$ :

$$\underline{\theta}_n^\varepsilon(\pi_n, t, x) := \inf_{b \in \mathcal{B}_{L_0}^t} \underline{\mathcal{E}}_t^{L_0} \left[ e^{\int_t^T b_r dr} \overline{\lim}_{i \rightarrow \infty} \xi \left( \omega^{\Pi_i^\varepsilon(\pi_n, t, x)} \right) + C_0 \int_t^T e^{\int_t^s b_r dr} ds \right].$$

*Step 2.* We now prove the lemma for  $G$ . Given the construction of Step 1, define:

$$\bar{\theta}_m^{\varepsilon,m}(\pi_m; t, x) := \bar{\theta}_m^\varepsilon(\pi_m; t, x), \quad \underline{\theta}_m^{\varepsilon,m}(\pi_m; t, x) := \underline{\theta}_m^\varepsilon(\pi_m; t, x); \quad m \geq 1.$$

By Proposition 3.2 (ii), for  $n = m-1, \dots, 0$ , we may define  $\bar{\theta}_n^{\varepsilon,m}$  and  $\underline{\theta}_n^{\varepsilon,m}$  as the unique viscosity solution of the PDE (E) $_{\varepsilon,0}^{t_n, \omega^{\pi_n}}$  with boundary conditions  $\bar{\theta}_n^{\varepsilon,m} = \bar{\theta}_{n+1}^{\varepsilon,m}$  and  $\underline{\theta}_n^{\varepsilon,m} = \underline{\theta}_{n+1}^{\varepsilon,m}$  on  $\partial \mathcal{O}_{t_n}^\varepsilon$ . Note that, for  $(t, x) \in \partial \mathcal{O}_{t_m}^\varepsilon$ ,

$$\bar{\theta}_m^{\varepsilon,m}(\pi_m; t, x) = \bar{\theta}_{m+1}^{\varepsilon,m+1}(\pi_m^{t,x}; t, 0), \quad \underline{\theta}_m^{\varepsilon,m}(\pi_m; t, x) = \underline{\theta}_{m+1}^{\varepsilon,m+1}(\pi_m^{t,x}; t, 0).$$

By the comparison result of Proposition 3.2 (i), we also have that

$$\bar{\theta}_m^{\varepsilon,m}(\pi_m; \cdot) \geq \bar{\theta}_m^{\varepsilon,m+1}(\pi_m; \cdot) \geq \underline{\theta}_m^{\varepsilon,m+1}(\pi_m; \cdot) \geq \underline{\theta}_m^{\varepsilon,m}(\pi_m; \cdot) \quad \text{in } \mathcal{O}_{t_m}^\varepsilon,$$

and therefore, by the same comparison argument:

$$\bar{\theta}_n^{\varepsilon,m}(\pi_n; \cdot) \geq \bar{\theta}_n^{\varepsilon,m+1}(\pi_n; \cdot) \geq \underline{\theta}_n^{\varepsilon,m+1}(\pi_n; \cdot) \geq \underline{\theta}_n^{\varepsilon,m}(\pi_n; \cdot) \quad \text{in } \mathcal{O}_{t_n}^\varepsilon, \text{ for all } n \leq m. \quad (5.10)$$

Denote  $\delta \theta_n^{\varepsilon,m} := \bar{\theta}_n^{\varepsilon,m} - \underline{\theta}_n^{\varepsilon,m}$ . For any  $\pi_n$  and any  $(t, x) \in \mathcal{O}_{t_n}^\varepsilon$ , recall the notations in (5.9). Applying Lemma 5.2 repeatedly, and following similar but much easier arguments as those in Steps 2 and 3 of Proposition 4.1, we see that:

$$|\delta \theta_n^{\varepsilon,m}(\pi_n; t, x)| \leq \bar{\mathcal{E}}_t^{L_0} \left[ \left| \delta \theta_m^{\varepsilon,m} \left( \Pi_{m-n}(\pi_n, t, x); \mathbf{H}_{m-n}, 0 \right) \right| \right].$$

Note that  $\delta\theta_n^{\varepsilon,m}(\pi_n; t, x) = 0$  when  $t = T$ . Then, by Lemma 2.5 again,

$$|\delta\theta_n^{\varepsilon,m}(\pi_n; t, x)| \leq CC_t^{L_0} \left[ H_{m-n} < T \right] \rightarrow 0 \text{ as } m \rightarrow \infty.$$

Together with (5.10), this implies the existence of  $\theta_n^\varepsilon$  such that

$$\bar{\theta}_n^{\varepsilon,m} \downarrow \theta_n^\varepsilon, \quad \underline{\theta}_n^{\varepsilon,m} \uparrow \theta_n^\varepsilon, \quad \text{as } m \rightarrow \infty. \quad (5.11)$$

Clearly  $\theta_n^\varepsilon$  are uniformly bounded and continuous. Finally, it follows from the stability of viscosity solutions that  $\theta_n^\varepsilon$  satisfies (5.7).  $\blacksquare$

We now define  $H_0^\varepsilon := 0$  and

$$H_{n+1}^\varepsilon := T \wedge \inf \{ t \geq H_n^\varepsilon : |B_t - B_{H_n^\varepsilon}| = \varepsilon \} \quad \text{for all } n \geq 0.$$

Then clearly  $H_n^\varepsilon$  satisfies the requirements of Definition 2.6. Let  $\hat{\pi}_n$  denote the sequence  $(H_i^\varepsilon, B_{H_i^\varepsilon})_{1 \leq i \leq n}$ , and  $\omega^\varepsilon := \lim_{n \rightarrow \infty} \omega^{\hat{\pi}_n}$ . It is clear that

$$\|\omega - \omega^\varepsilon\|_T \leq \varepsilon \quad \text{and} \quad \|\omega^{\hat{\pi}_n} - \omega\|_t \leq 2\varepsilon, \quad \text{for all } t, n, \omega. \quad (5.12)$$

**Lemma 5.4** *Assume Assumptions 2.7, 3.1, and 3.3. Then there exists  $\psi^\varepsilon \in \bar{C}^{1,2}(\Lambda)$  bounded from below with corresponding stopping times  $H_n^\varepsilon$  such that*

$$\psi^\varepsilon(0, \mathbf{0}) = \theta_0^\varepsilon(0, \mathbf{0}) + \varepsilon, \quad \psi^\varepsilon(T, \omega) \geq \xi(\omega^\varepsilon), \quad \mathbf{L}^{H_n^\varepsilon, \omega^{\hat{\pi}_n}} \psi^\varepsilon \geq 0 \text{ on } [H_n^\varepsilon, H_{n+1}^\varepsilon). \quad (5.13)$$

**Proof** For notational simplicity, in this proof we omit the superscript  $\varepsilon$  and denote  $\theta = \theta^\varepsilon$ ,  $\psi = \psi^\varepsilon$  etc. Set  $\delta_n := 2^{-n-2}\varepsilon$ . We prove the lemma in two steps.

*Step 1.* First, by applying Assumption 3.1 to PDE (E) $_{\varepsilon,0}^{t_0, \omega^{\pi_0}}$ , there exists a function  $v_0 \in C^{1,2}(\bar{\mathcal{O}}_0^\varepsilon)$  such that

$$v_0(0, \mathbf{0}) = \theta_0(0, \mathbf{0}) + \frac{\varepsilon}{2}, \quad \mathbf{L}^{0,0} v_0 \geq 0 \text{ in } \mathcal{O}_0^\varepsilon, \quad v_0 \geq \theta_0 \text{ on } \partial \mathcal{O}_0^\varepsilon.$$

Set

$$\psi(t, \omega) := v_0(t, \omega_t) + \sum_{i \geq 0} \delta_i, \quad t \in [0, H_1]. \quad (5.14)$$

By the monotonicity (4.1), it is clear that

$$\psi^\varepsilon(0, \mathbf{0}) - \theta_0^\varepsilon(0, \mathbf{0}) = \frac{\varepsilon}{2} + \sum_{i \geq 0} \delta_i = \varepsilon, \quad \text{and} \quad \mathbf{L}^{0,0} \psi \geq \mathbf{L}^{0,0} v_0 \geq 0, \quad t \in [0, H_1).$$

When  $H_1 < T$ , we have

$$v_0(H_1, \omega_{H_1}) \geq \theta_0(H_1, \omega_{H_1}) = \theta_1(\hat{\pi}_1; H_1, 0),$$

Moreover, it is clear that  $\psi \in C^{1,2}(\bar{\Lambda}(H_1))$ ,  $\psi, \partial_t \psi, \partial_\omega \psi, \partial_{\omega\omega}^2 \psi$  are all bounded on  $[0, H_1]$ , and  $\psi(H_1, \omega)$  satisfies (2.11) with  $E_1^0 = \Omega$ ,  $\varphi_1^0 = \sum_{i \geq 0} \delta_i$ ,  $t \in [0, H_1]$ , and  $\psi_1^0(t, \omega) = v_0(t, \omega_t)$ .

Next, for each  $\pi_1 := (t_1, x_1) \in \partial \mathcal{O}_0^\varepsilon$ , applying Assumption 3.1 to PDE (E) $_{\varepsilon,0}^{t_1, \omega^{\pi_1}}$ , there exists a function  $v_1(\pi_1; \cdot) \in C^{1,2}(\bar{\mathcal{O}}_{t_1}^\varepsilon)$  such that

$$v_1(\pi_1; t_1, 0) = v_0(\pi_1) + \delta_0, \quad \mathbf{L}^{t_1, \omega^{\pi_1}} v_1 \geq 0 \text{ in } \mathcal{O}_{t_1}^\varepsilon, \quad v_1(\pi_1; \cdot) \geq \theta_1(\pi_1; \cdot) \text{ on } \partial \mathcal{O}_{t_1}^\varepsilon, \quad (5.15)$$

Set

$$\psi(t, \omega) := v_1(\hat{\pi}_1; t, \omega_t - \omega_{H_1}) + \sum_{i \geq 1} \delta_i, \quad t \in (H_1, H_2]. \quad (5.16)$$

It is clear that

$$\psi^{H_1(\omega), \omega} \in C^{1,2}(\bar{\Lambda}^{H_1(\omega)}(H_2^{H_1(\omega), \omega})) \quad \text{and} \quad \psi, \partial_t \psi, \partial_\omega \psi, \partial_{\omega\omega}^2 \psi \text{ are bounded in } [H_1, H_2].$$

We shall prove in Step 2 below that

$$\text{we may construct } v_1 \text{ so that } v_1 \text{ satisfies (5.15) and } \psi(H_2, \omega) \text{ satisfies (2.11).} \quad (5.17)$$

Repeating the above arguments we may construct a sequence of functions  $v_n$  and thus construct  $\psi \in \bar{\mathcal{C}}^{1,2}(\Lambda)$  satisfying (5.13). Finally, by Lemma 5.3,  $\theta_n \geq -C$ . By applying the comparison principle Proposition 4.1 we see that  $v_n \geq -C$ , and thus  $\psi \geq -C$ .

*Step 2.* We now prove (5.17). Let  $\eta > 0$ ,  $\lambda > 0$  be small numbers which will be decided later. Set  $s_i := (1 - \eta)^i T$ ,  $i \geq 0$ . Since  $\partial \mathcal{O}_0^\varepsilon$  is compact, there exist a partition  $D_1, \dots, D_n$  of  $\{y : |y| = \varepsilon\}$  such that  $|y - \tilde{y}| \leq \eta$  for any  $y, \tilde{y} \in D_j$ ,  $j = 1, \dots, n$ . For each  $j$ , fix a point  $y_j \in D_j$ . Now for each  $(i, j)$ , thanks to Assumption 3.1 let  $w_{ij}^\lambda$  denote the unique viscosity solution of the PDE (E) $_{\varepsilon, \eta}^{s_i, \omega^{s_i, y_j}}$  with boundary condition  $w_{ij}^\lambda(t, x) = \theta_1(s_i, y_j; t \wedge T, x) + \lambda$  on  $\partial \mathcal{O}_{s_i}^{\varepsilon, \eta}$ . Then by standard arguments there exist  $\eta_0(\lambda)$  and  $C_0(\lambda)$ , which may depend on  $L_0$ ,  $\lambda$  and the regularity of  $\theta_1$ , such that, for all  $\eta \leq \eta_0(\lambda)$ ,

$$0 \leq w_{ij}^\lambda(t, x) - \theta_1(s_i, y_j; t \wedge T, x) \leq C_0(\lambda) \quad \text{on } \bar{\mathcal{O}}_{s_i}^{\varepsilon, \eta} \setminus \mathcal{O}_{s_i}^\varepsilon.$$

In particular, the above inequalities hold on  $\partial \mathcal{O}_{s_i}^\varepsilon$ . Then, by the comparison principle Proposition 3.2 (i) and Lemma 5.2, we have

$$0 \leq w_{ij}^\lambda(t, x) - \theta_1(s_i, y_j; t \wedge T, x) \leq C_0(\lambda) \quad \text{in } \bar{\mathcal{O}}_{s_i}^{\varepsilon, \eta}. \quad (5.18)$$

It is clear that  $\lim_{\lambda \rightarrow 0} C_0(\lambda) = 0$ . Fix  $\lambda_0$  such that  $C_0(\lambda_0) < \frac{\delta_0}{2}$ . Then

$$w_{ij}^\lambda(s_i, 0) < \theta_1(s_i, y_j; s_i, 0) + \frac{\delta_0}{2} = \theta_0(s_i, y_j) + \frac{\delta_0}{2} \leq v_0(s_i, y_j) + \frac{\delta_0}{2}.$$

Now for  $\eta \leq \eta_0(\lambda_0)$ , by Assumption 3.1 there exists  $w_{ij} \in C^{1,2}(\mathcal{O}_{s_i}^{\varepsilon, \eta})$  satisfying

$$w_{ij}(s_i, 0) = v_0(s_i, y_j) + \frac{\delta_0}{2}, \quad \mathbf{L}^{s_i, \omega^{(s_i, y_j)}} w_{ij} \geq 0 \text{ in } \mathcal{O}_{s_i}^{\varepsilon, \eta}, \quad w_{ij} \geq w_{ij}^{\lambda_0} \text{ on } \partial \mathcal{O}_{s_i}^{\varepsilon, \eta}. \quad (5.19)$$

We note that, by the comparison principle Proposition 3.2 (i) again we have

$$w_{ij}(t, x) \geq w_{ij}^{\lambda_0}(t, x) \geq \theta_1(s_i, y_j; t \wedge T, x) \text{ on } \overline{\mathcal{O}}_{s_i}^{\varepsilon, \eta}. \quad (5.20)$$

Now for any  $\pi_1 := (t_1, x_1) \in \partial \mathcal{O}_0^\varepsilon$ , there exists unique  $(i, j)$  such that  $s_{i+1} < t_1 \leq s_i$  and  $x_1 \in D_j$ . Define

$$\begin{aligned} v_1(\pi_1; t, x) &:= v_0(\pi_1) + \frac{\delta_0}{2} \\ &+ \sum_{i,j} \mathbf{1}_{(s_{i+1}, s_i] \times D_j}(t_1, x_1) (w_{ij}(s_i - t_1 + t, x) - v_0(s_i, y_j)), \quad (t, x) \in \overline{\mathcal{O}}_{t_1}^\varepsilon. \end{aligned} \quad (5.21)$$

Clearly the above  $v_1$  is measurable in all variables  $(t_1, x_1, t, x)$  and, by (5.19),

$$v_1(\pi_1; t_1, 0) = v_0(\pi_1) + \delta_0.$$

By choosing  $\eta$  small enough, we may have

$$|v_0(s_i, y_j) - v_0(\pi_1)| \leq \frac{\delta_0}{4} \quad \text{and} \quad \rho_0\left(\eta(1 + T + \varepsilon)\right) \leq \frac{\delta_0}{4}. \quad (5.22)$$

Then

$$v_1(\pi_1; t, x) \geq w_{ij}(s_i - t_1 + t, x) + \frac{\delta_0}{4}. \quad (5.23)$$

Denote by  $(i, j)$  the unique pair of integers such that  $(t_1, x_1) \in (s_{i+1}, s_i] \times D_j$ , and  $\tilde{t} := s_i - t_1 + t$ . By (4.1) we see that

$$\begin{aligned} \mathbf{L}^{t_1, \omega^{\pi_1}} v_1(\pi_1; t, x) &\geq \mathbf{L}^{t_1, \omega^{t_1}} v_1(\pi_1; t, x) - \mathbf{L}^{s_i, \omega^{(s_i, y_j)}} w_{ij}(\tilde{t}, x) \\ &= -G(t, \omega_{\wedge t_1}^{\pi_1}, v_1(\pi_1, t, x), Dw_{ij}(\tilde{t}, x), D^2 w_{ij}(\tilde{t}, x)) \\ &\quad + G(\tilde{t} \wedge T, \omega_{\wedge s_i}^{(s_i, y_j)}, w_{ij}(\tilde{t}, x), Dw_{ij}(\tilde{t}, x), D^2 w_{ij}(\tilde{t}, x)) \\ &\geq \frac{\delta_0}{4} - G(t, \omega_{\wedge t_1}^{\pi_1}, w_{ij}(\tilde{t}, x), Dw_{ij}(\tilde{t}, x), D^2 w_{ij}(\tilde{t}, x)) \\ &\quad + G(\tilde{t} \wedge T, \omega_{\wedge s_i}^{(s_i, y_j)}, w_{ij}(\tilde{t}, x), Dw_{ij}(\tilde{t}, x), D^2 w_{ij}(\tilde{t}, x)) \\ &\geq \frac{\delta_0}{4} - \rho_0\left(\mathbf{d}_\infty((t, \omega_{\wedge t_1}^{\pi_1}), (\tilde{t} \wedge T, \omega_{\wedge s_i}^{(s_i, y_j)}))\right). \end{aligned}$$

Observing that  $0 \leq s_i - t_1 \leq s_i - s_{i+1} = \eta s_i \leq \eta T$ , it follows that:

$$\begin{aligned}
& \mathbf{d}_\infty((t, \omega_{\cdot \wedge t_1}^{\pi_1}), (\tilde{t} \wedge T, \omega_{\cdot \wedge s_i}^{(s_i, y_j)})) \\
& \leq |t - \tilde{t}| + \sup_{0 \leq s \leq T} \left| \frac{s \wedge t_1}{t_1} x_1 - \frac{s \wedge s_i}{s_i} y_j \right| \\
& \leq \eta T + \sup_{0 \leq s \leq T} \left| \frac{s \wedge t_1}{t_1} x_1 - \frac{s \wedge t_1}{t_1} y_j \right| + \sup_{0 \leq s \leq T} \left| \frac{s \wedge t_1}{t_1} y_j - \frac{s \wedge s_i}{s_i} y_j \right| \\
& \leq \eta(1 + T) + \varepsilon \sup_{0 \leq s \leq T} \left| \frac{s \wedge t_1}{t_1} - \frac{s \wedge s_i}{s_i} \right| \\
& = \eta(1 + T) + \varepsilon \left[ 1 - \frac{t_1}{s_i} \right] \leq \eta(1 + T) + \varepsilon \left[ 1 - \frac{s_{i+1}}{s_i} \right] = \eta(1 + T + \varepsilon).
\end{aligned}$$

This, together with (5.22), implies that

$$\mathbf{L}^{t_1, \omega^{\pi_1}} v_1(\pi_1; t, x) \geq \frac{\delta_0}{4} - \rho_0(\eta(1 + T + \varepsilon)) \geq 0.$$

Moreover, by (5.23) and (5.20) we have

$$(v_1 - \theta_1)(\pi_1; t, x) \geq \frac{\delta_0}{4} + \theta_1(s_i, y_j; (s_i - t_1 + t) \wedge T, x) - \theta_1(\pi_1; t, x),$$

which is positive when  $\eta$  is small enough. This proves that  $v_1$  satisfies (5.15).

Finally, recall (5.16) and (5.21). For each  $(i, j)$ , define

$$\begin{aligned}
E_{i,j}^1 &:= \{s_{i+1} < H_1 \leq s_i\} \cap \{B_{H_1} \in D_j\}, \\
\varphi_{i,j}^1(t, \omega) &:= v_0(t, \omega) + \frac{\delta_0}{2} - v_0(s_i, y_j), \quad \psi_{i,j}^1(t, \omega) := w_{ij}(s_i + t, x).
\end{aligned}$$

Here we are using  $(i, j)$  instead of  $j$  as index. Then it is straightforward to check that  $\psi(H_2, \omega)$  satisfies (2.11). ■

**Proof of Proposition 5.1.** For any  $\varepsilon > 0$ , let  $H_n^\varepsilon$ ,  $n \geq 0$ , and  $\psi^\varepsilon$  be as in Lemma 5.4, and define  $\overline{\psi}^\varepsilon := \psi^\varepsilon + \rho_0(2\varepsilon)$ . Then clearly  $\overline{\psi}^\varepsilon \in \overline{C}^{1,2}(\Lambda)$ ,  $\overline{\psi}^\varepsilon$  is bounded from below, and

$$\overline{\psi}^\varepsilon(T, \omega) - \xi(\omega) = \psi^\varepsilon(T, \omega) + \rho_0(2\varepsilon) - \xi(\omega) \geq \xi(\omega^\varepsilon) - \xi(\omega) + \rho_0(2\varepsilon) \geq 0.$$

where the last inequality thanks to (5.12). Moreover, for  $H_n \leq t < H_{n+1}$ , by (4.1) we have

$$\begin{aligned}
\mathcal{L}\overline{\psi}^\varepsilon(t, \omega) &= -\partial_t \psi^\varepsilon(t, \omega) - G(t, \omega, \psi^\varepsilon + \rho_0(2\varepsilon), \partial_\omega \psi^\varepsilon, \partial_{\omega\omega}^2 \psi^\varepsilon) \\
&\geq G(t, \omega_{\cdot \wedge H_n}^{\hat{\pi}_n}, \psi^\varepsilon, \partial_\omega \psi^\varepsilon, \partial_{\omega\omega}^2 \psi^\varepsilon) - G(t, \omega, \psi^\varepsilon + \rho_0(2\varepsilon), \partial_\omega \psi^\varepsilon, \partial_{\omega\omega}^2 \psi^\varepsilon) \\
&\geq \rho_0(2\varepsilon) + G(t, \omega_{\cdot \wedge H_n}^{\hat{\pi}_n}, \psi^\varepsilon, \partial_\omega \psi^\varepsilon, \partial_{\omega\omega}^2 \psi^\varepsilon) - G(t, \omega, \psi^\varepsilon, \partial_\omega \psi^\varepsilon, \partial_{\omega\omega}^2 \psi^\varepsilon) \\
&\geq \rho_0(2\varepsilon) - \rho_0\left(\|\omega_{\cdot \wedge H_n}^{\hat{\pi}_n} - \omega\|_t\right) \geq 0.
\end{aligned}$$

Then by the definition of  $\bar{u}$  we see that

$$\bar{u}(0, \mathbf{0}) \leq \bar{\psi}^\varepsilon(0, \mathbf{0}) = \psi^\varepsilon(0, \mathbf{0}) + \rho_0(2\varepsilon) \leq \theta_0^\varepsilon(0, \mathbf{0}) + \varepsilon + \rho_0(2\varepsilon).$$

Similarly,  $\underline{u}(0, \mathbf{0}) \geq \theta_0^\varepsilon(0, \mathbf{0}) - \varepsilon - \rho_0(2\varepsilon)$ . This implies that

$$\bar{u}(0, \mathbf{0}) - \underline{u}(0, \mathbf{0}) \leq 2(\varepsilon + \rho_0(2\varepsilon)).$$

Since  $\varepsilon > 0$  is arbitrary, we prove that  $\bar{u}(0, \mathbf{0}) = \underline{u}(0, \mathbf{0})$ . Similarly we can show that  $\bar{u}(t, \omega) = \underline{u}(t, \omega)$  for all  $(t, \omega) \in \Lambda$ . ■

**Remark 5.5** In some special cases where a candidate solution of the PPDE is available through some representation, Proposition 5.1 can be proved by a much easier argument. See [7] and [5] in the context of a semilinear PPDE.

## 6 Construction of a viscosity solution

By the previous Proposition 5.1, we have  $\bar{u} = \underline{u} =: u$ . In this section, we show that  $u$  is a viscosity solution of our PPDE. We first verify that  $u$  has the required regularity.

**Proposition 6.1** *Under Assumptions 2.7 and 3.3, we have  $\bar{u} \in \bar{\mathcal{U}}$  and  $\underline{u} \in \underline{\mathcal{U}}$ .*

**Proof** We only prove the claim for  $\bar{u}$ . The proof for  $\underline{u}$  is similar.

(i) We first show that  $\bar{u}$  and  $\underline{u}$  are bounded. Fix  $(t, \omega)$ , and set

$$\psi(s, \tilde{\omega}) := C_0(L_0 + 1)e^{(L_0+1)(T-s)}.$$

Then  $\psi \in \bar{C}^{1,2}(\Lambda^t)$ ,  $\psi_T \geq C_0(L_0 + 1) \geq C_0 \geq \xi^{t, \omega}$ , and we compute that

$$(\mathcal{L}\psi)_s^{t, \omega} = (L_0 + 1)\psi_s - G^{t, \omega}(\cdot, \psi_s, \mathbf{0}, \mathbf{0}) \geq \psi_s - G^{t, \omega}(\cdot, \mathbf{0}, \mathbf{0}, \mathbf{0}) \geq C_0(L_0 + 1) - C_0 \geq 0.$$

This implies that  $\psi \in \bar{\mathcal{D}}(t, \omega)$ , and thus  $\bar{u}(t, \omega) \leq \psi(t, \mathbf{0}) = C_0(L_0 + 1)e^{(L_0+1)(T-t)}$ . Similarly one can show that  $\underline{u}(t, \omega) \geq -C_0(L_0 + 1)e^{(L_0+1)(T-t)}$ . By (5.3), we obtain

$$-C_0(L_0 + 1)e^{(L_0+1)(T-t)} \leq \underline{u} \leq \bar{u} \leq C_0(L_0 + 1)e^{(L_0+1)(T-t)}.$$

(ii) We next show that  $\bar{u}$  is uniformly continuous in  $\omega$ , uniformly in  $t$ . For  $t \in [0, T]$  and  $\omega^1, \omega^2 \in \Omega$ , and  $\psi^1 \in \bar{\mathcal{D}}(t, \omega^1)$ , denote  $\delta := \|\omega^1 - \omega^2\|_t$  and introduce the process

$$\psi^2(s, \tilde{\omega}) := \psi^1(s, \tilde{\omega}) + (L_0 + 1)e^{(L_0+1)(T-s)}[\rho_0(\delta) + \delta].$$



Clearly  $\psi^2 \in \bar{C}^{1,2}(\Lambda^t)$ , and

$$\psi_T^2 = \psi_T^1 + (L_0 + 1) \left[ \rho_0(\delta) + \delta \right] \geq \xi_T^{t, \omega^1} + \rho_0(\delta) \geq \xi^{t, \omega^2}.$$

Moreover,

$$\begin{aligned} (\mathcal{L}\psi^2)_s^{t, \omega^2} &\geq (\mathcal{L}\psi^2)_s^{t, \omega^2} - (\mathcal{L}\psi^1)_s^{t, \omega^1} \\ &\geq (L_0 + 1)^2 \left[ \rho_0(\delta) + \delta \right] - G^{t, \omega^2}(\cdot, \psi^2, \partial_\omega \psi^1, \partial_{\omega\omega}^2 \psi^1) + G^{t, \omega^1}(\cdot, \psi^1, \partial_\omega \psi^1, \partial_{\omega\omega}^2 \psi^1) \\ &\geq (L_0 + 1)^2 \left[ \rho_0(\delta) + \delta \right] - L_0(\delta + |\psi^2 - \psi^1|) = [L_0 + 1]\rho_0(\delta) + \delta > 0. \end{aligned}$$

Then  $\psi^2 \in \bar{\mathcal{D}}(t, \omega^2)$ , and therefore  $\bar{u}(t, \omega^2) \leq \psi^2(t, \mathbf{0})$ , implying that

$$\bar{u}(t, \omega^2) - \psi^1(t, \mathbf{0}) \leq \psi^2(t, \mathbf{0}) - \psi^1(t, \mathbf{0}) = (L_0 + 1)e^{(L_0+1)(T-t)}[\rho_0(\delta) + \delta] \leq C[\rho_0(\delta) + \delta].$$

Since  $\psi^1 \in \bar{\mathcal{D}}(t, \omega^1)$  is arbitrary, we obtain  $\bar{u}(t, \omega^2) - \bar{u}(t, \omega^1) \leq C\delta$ . By symmetry, this shows the required uniform continuity of  $\bar{u}$  in  $\omega$ , uniformly in  $t$ .

(iii) We now prove that  $-\bar{u}$  satisfies (2.2). For  $t_1 < t_2$  and  $\omega \in \Omega$ , fix some  $\psi^2 \in \bar{\mathcal{D}}(t_2, \omega)$ , and consider the following PPDE on  $[t_1, t_2]$ :

$$\begin{aligned} -\partial_t u_0 - C_0 - L_0|u_0| - G_0(\partial_\omega u_0, \partial_{\omega\omega}^2 u_0) &= 0, \quad t \in [t_1, t_2], \tilde{\omega} \in \Omega^{t_1}, \\ u_0(t_2, \tilde{\omega}) &= \psi^2(t_2, \mathbf{0}) + e^{L_0(T-t_2)} \rho_0 \left( \mathbf{d}_\infty((t_1, \omega), (t_2, \omega)) + \|\tilde{\omega}\|_{t_2} \right), \end{aligned} \quad (6.1)$$

where  $G_0(z, \gamma) := \sup_{|\alpha| \leq L_0, \beta \leq \sqrt{2L_0}I_d} \{\alpha \cdot z + \frac{1}{2}\beta^2 : \gamma\}$ . This PPDE clearly satisfies Assumptions 2.7, 2.8, and 3.3 (with  $T = t_2$ ). We shall prove in Proposition 7.2 (i) below that it also satisfies Assumption 3.1. Then, it follows from the comparison result of Theorem 3.4 (i), which we proved in the previous section, that the PPDE (6.1) satisfies the comparison result. Next, in our accompanying paper [7] Proposition 4.8, the solution  $u_0$  of PPDE (6.1) is constructed by means of a convenient stochastic control representation (or equivalently, a second order backward SDE), and satisfies:

$$\begin{aligned} |u_0(t_1, \mathbf{0}) - \psi^2(t_2, \mathbf{0})| &\leq C\bar{\mathcal{E}}_{t_1}^{L_0} \left[ \rho_0 \left( \mathbf{d}_\infty((t_1, \omega), (t_2, \omega)) + \|B^{t_1}\|_{t_2} \right) \right] \\ &\leq C\rho \left( \mathbf{d}_\infty((t_1, \omega), (t_2, \omega)) \right), \end{aligned} \quad (6.2)$$

for some modulus of continuity  $\rho$ . Moreover, by the comparison result together with Proposition 5.1, which we proved in the previous section, this unique viscosity solution of PPDE (6.1) satisfies  $u_0 = \bar{u}_0$ . Then for any  $\varepsilon > 0$ , there exists  $\psi^1 \in \bar{C}^{1,2}(\Lambda^{t_1})$  such that

$$\begin{aligned} \psi^1(t_1, \mathbf{0}) &\leq u_0(t_1, \mathbf{0}) + \varepsilon, \quad \psi^1(t_2, \tilde{\omega}) \geq u_0(t_2, \tilde{\omega}); \\ -\partial_t \psi^1 - C_0 - L_0|\psi^1| - G_0(\partial_\omega \psi^1, \partial_{\omega\omega}^2 \psi^1) &\geq 0. \end{aligned} \quad (6.3)$$

Therefore, for  $t \in [t_1, t_2)$ ,

$$\begin{aligned}
\mathcal{L}\psi^1 &= -\partial_t \psi^1 - G(\cdot, \psi^1, \partial_\omega \psi^1, \partial_{\omega\omega}^2 \psi^1) \\
&\geq C_0 + L_0 |\psi^1| + G_0(\partial_\omega \psi^1, \partial_{\omega\omega}^2 \psi^1) - G(\cdot, \psi^1, \partial_\omega \psi^1, \partial_{\omega\omega}^2 \psi^1) \\
&\geq L_0 |\psi^1| + G_0(\partial_\omega \psi^1, \partial_{\omega\omega}^2 \psi^1) - G(\cdot, \psi^1, \partial_\omega \psi^1, \partial_{\omega\omega}^2 \psi^1) + G(\cdot, 0, \mathbf{0}, \mathbf{0}) \geq 0, \quad (6.4)
\end{aligned}$$

by Assumption 2.7 and the definition of  $G_0$ .

Now extend  $\psi^1$  to  $[t_2, T]$  by

$$\psi^1(t, \tilde{\omega}) := \psi^2(t, \tilde{\omega}^{t_2}) + \left( \psi^1(t_2, \tilde{\omega}) - \psi^2(t_2, \mathbf{0}) \right) e^{L_0(t_2-t)}. \quad (6.5)$$

It is straightforward to check that  $\psi^1 \in \bar{C}^{1,2}(\Lambda^{t_1})$ . It follows from (6.1) and (6.3) that  $\psi^1(t_2, \tilde{\omega}) \geq u_0(t_2, \tilde{\omega}) \geq \psi^2(t_2, \mathbf{0})$ , and thus  $\psi^1(t, \tilde{\omega}) \geq \psi^2(t, \tilde{\omega}^{t_2})$ . Then by (4.1) we have, for  $t \in [t_2, T]$ ,

$$\begin{aligned}
-\mathcal{L}\psi^1 &= -\partial_t \psi^2 + L_0(\psi^1 - \psi^2(t, \tilde{\omega}^{t_2})) - G(\cdot, \psi^1, \partial_\omega \psi^2, \partial_{\omega\omega}^2 \psi^2) \\
&\geq L_0(\psi^1 - \psi^2(t, \tilde{\omega}^{t_2})) + G(\cdot, \psi^2, \partial_\omega \psi^2, \partial_{\omega\omega}^2 \psi^2) - G(\cdot, \psi^1, \partial_\omega \psi^2, \partial_{\omega\omega}^2 \psi^2) \geq 0.
\end{aligned} \quad (6.6)$$

Moreover, by (6.5), (6.3), and (6.1),

$$\begin{aligned}
\psi^1(T, \tilde{\omega}) &\geq \psi^2(T, \tilde{\omega}^{t_2}) + (u_0(t_2, \tilde{\omega}) - \psi^2(t_2, \mathbf{0})) e^{L_0(t_2-T)} \\
&\geq \xi^{t_2, \omega}(\tilde{\omega}^{t_2}) + \rho_0(\mathbf{d}_\infty((t_1, \omega), (t_2, \omega)) + \|\tilde{\omega}\|_{t_2}) \geq \xi^{t_1, \omega}(\tilde{\omega}).
\end{aligned}$$

This, together with (6.4) and (6.6), implies that  $\psi^1 \in \overline{\mathcal{D}}(t_1, \omega)$ . Then it follows from (6.3) and (6.2) that

$$\overline{u}(t_1, \omega) \leq \psi^1(t_1, \mathbf{0}) \leq u_0(t_1, \mathbf{0}) + \varepsilon \leq \psi^2(t_2, \mathbf{0}) + C\rho(\mathbf{d}_\infty((t_1, \omega), (t_2, \omega))) + \varepsilon.$$

Since  $\psi^2 \in \overline{\mathcal{D}}(t_2, \omega)$  and  $\varepsilon > 0$  are arbitrary, this provides (2.2).

(iv) By Remark 3.2 in [6], it remains only to prove  $\overline{u}$  is right continuous in  $t$ . By (iii), it is clear that  $\lim_{s \downarrow t} \overline{u}(s, \omega) \geq \overline{u}(t, \omega)$ . On the other hand, for any  $\varepsilon > 0$ , there exists  $\psi \in \overline{\mathcal{D}}(t, \omega)$  such that  $\psi(t, \mathbf{0}) \leq \overline{u}(t, \omega) + \varepsilon$ . Clearly  $\psi(s, \omega^t) \geq \overline{u}(s, \omega)$ . Since  $\psi$  is continuous, we obtain

$$\overline{\lim}_{s \downarrow t} \overline{u}(s, \omega) \leq \overline{\lim}_{s \downarrow t} \psi(s, \omega^t) = \psi(t, \mathbf{0}) \leq \overline{u}(t, \omega) + \varepsilon.$$

Since  $\varepsilon$  is arbitrary, this completes the proof. ■

**Proposition 6.2** *Under Assumptions 2.7 and 3.3,  $\overline{u}$  (resp.  $\underline{u}$ ) is a viscosity  $L_0$ -supersolution (resp.  $L_0$ -subsolution) of PPDE (2.12) with terminal condition  $\xi$ .*

**Proof** Without loss of generality, we may assume that the generator  $G$  satisfies (4.1), and we prove only that  $\bar{u}$  is a viscosity  $L_0$ -supersolution, and we check only its viscosity property at  $t = 0$ .

First, for any  $\psi \in \bar{\mathcal{D}}(0, \mathbf{0})$  and any  $(t, \omega) \in \Lambda$ , it is clear that  $\psi^{t, \omega} \in \bar{\mathcal{D}}(t, \omega)$  and then  $\psi(t, \omega) \geq \bar{u}(t, \omega)$ . Thus we have the following partial Dynamic Programming Principle:

$$\bar{u}(0, \mathbf{0}) \geq \inf \left\{ \psi(0, \mathbf{0}) : \psi \in \bar{C}^{1,2}(\Lambda), (\mathcal{L}\psi)_s \geq 0, s \in [0, \tau] \text{ and } \psi_\tau \geq \bar{u}_\tau \right\} \text{ for } \tau \in \mathcal{T}. \quad (6.7)$$

Now assume  $\bar{u}$  does not satisfy the viscosity  $L_0$ -superproperty at  $(0, \mathbf{0})$ . Then by [7] Remark 3.13 (i), there exists  $\varphi \in \bar{\mathcal{A}}^{L_0} \bar{u}(0, \mathbf{0})$  such that  $\varphi(0, \mathbf{0}) = \bar{u}(0, \mathbf{0})$  and  $-c := \mathcal{L}\varphi(0, \mathbf{0}) < 0$ . Let  $\mathbb{H}$  be the stopping time required in  $\bar{\mathcal{A}}^{L_0} \bar{u}(0, \mathbf{0})$ . By (6.7), there exists  $\psi^n \in \bar{C}^{1,2}(\Lambda)$  such that

$$\psi^n(0, \mathbf{0}) \downarrow \bar{u}(0, \mathbf{0}) \text{ as } n \rightarrow \infty, \quad (\mathcal{L}\psi^n)_s \geq 0, s \in [0, \mathbb{H}] \text{ and } \psi^n_{\mathbb{H}} \geq \bar{u}_{\mathbb{H}}. \quad (6.8)$$

Since  $\varphi \in \bar{\mathcal{A}}^{L_0} \bar{u}(0, \mathbf{0})$ , this implies for all  $\mathbb{P} \in \mathcal{P}_{L_0}$  and  $n$  that:

$$0 \geq \mathbb{E}^{\mathbb{P}}[(\varphi - \bar{u})_{\mathbb{H}}] \geq \mathbb{E}^{\mathbb{P}}[(\varphi - \psi^n)_{\mathbb{H}}]. \quad (6.9)$$

Since  $\varphi \in C^{1,2}(\Lambda)$ , without loss of generality we may assume  $\mathcal{L}\varphi(t, \omega) \leq -\frac{c}{2}$  for all  $t \leq \mathbb{H}$ . Recall (2.4) and denote  $\mathcal{G}^{\mathbb{P}}\phi := \alpha^{\mathbb{P}} \cdot \partial_{\omega}\phi + \frac{1}{2}(\beta^{\mathbb{P}})^2 : \partial_{\omega\omega}^2\phi$ . Then, denoting  $\delta_n := (\psi^n - \bar{u})_0$  and applying Itô's formula in (6.9), it follows from (6.8) that:

$$\begin{aligned} \delta_n &\geq \mathbb{E}^{\mathbb{P}}[\psi_0^n - \psi_{\mathbb{H}}^n + \varphi_{\mathbb{H}} - \varphi_0] = \mathbb{E}^{\mathbb{P}}\left[\int_0^{\mathbb{H}} (\partial_t + \mathcal{G}^{\mathbb{P}})(\varphi - \psi^n)ds\right] \\ &\geq \mathbb{E}^{\mathbb{P}}\left[\int_0^{\mathbb{H}} \left(\frac{c}{2} - G(\cdot, \varphi, \partial_{\omega}\varphi, \partial_{\omega\omega}^2\varphi) + G(\cdot, \psi^n, \partial_{\omega}\psi^n, \partial_{\omega\omega}^2\psi^n) + \mathcal{G}^{\mathbb{P}}(\varphi - \psi^n)\right)ds\right] \\ &\geq \mathbb{E}^{\mathbb{P}}\left[\int_0^{\mathbb{H}} \left(\frac{c}{2} - G(\cdot, \varphi, \partial_{\omega}\varphi, \partial_{\omega\omega}^2\varphi) + G(\cdot, \bar{u}, \partial_{\omega}\psi^n, \partial_{\omega\omega}^2\psi^n) + \mathcal{G}^{\mathbb{P}}(\varphi - \psi^n)\right)ds\right], \end{aligned}$$

where the last inequality follows from (4.1) and the fact that  $\bar{u} \leq \psi^n$  by (6.8). Since  $\varphi_0 = \bar{u}_0$ ,  $\varphi, \bar{u} \in C^0(\Lambda)$ , and  $G$  is uniformly continuous in  $y$ , for possibly smaller  $\mathbb{H}$ :

$$\delta_n \geq \mathbb{E}^{\mathbb{P}}\left[\int_0^{\mathbb{H}} \left(\frac{c}{3} - G(\cdot, \bar{u}_0, \partial_{\omega}\varphi, \partial_{\omega\omega}^2\varphi) + G(\cdot, \bar{u}_0, \partial_{\omega}\psi^n, \partial_{\omega\omega}^2\psi^n) + \mathcal{G}^{\mathbb{P}}(\varphi - \psi^n)\right)ds\right].$$

We emphasize that the above  $\mathbb{H}$  is independent of  $n$ . Now let  $\eta > 0$  be a small number. For each  $n$ , define  $\tau_0^n := 0$ , and

$$\begin{aligned} \tau_{i+1}^n &:= \mathbb{H} \wedge \inf \left\{ t \geq \tau_i^n : \rho_0(\mathbf{d}_{\infty}((t, \omega), (\tau_i^n, \omega))) + |\partial_{\omega}\varphi(t, \omega) - \partial_{\omega}\varphi(\tau_i^n, \omega)| \right. \\ &\quad \left. + |\partial_{\omega\omega}^2\varphi(t, \omega) - \partial_{\omega\omega}^2\varphi(\tau_i^n, \omega)| + |\partial_{\omega}\psi^n(t, \omega) - \partial_{\omega}\psi^n(\tau_i^n, \omega)| \right. \\ &\quad \left. + |\partial_{\omega\omega}^2\psi^n(t, \omega) - \partial_{\omega\omega}^2\psi^n(\tau_i^n, \omega)| \geq \eta \right\}. \end{aligned}$$

By the smoothness of  $G$ ,  $\varphi$ , and  $\psi^n$ , one can easily check that  $\tau_i^n \uparrow \mathbf{H}$  as  $i \rightarrow \infty$ . Then

$$\begin{aligned} \delta_n \geq & \left[ \frac{c}{3} - C\eta \right] \mathbb{E}^{\mathbb{P}}[\mathbf{H}] \\ & + \sum_{i \geq 0} \mathbb{E}^{\mathbb{P}} \left[ (\tau_{i+1}^n - \tau_i^n) \left\{ G(\cdot, \bar{u}_0, \partial_\omega \psi^n, \partial_{\omega\omega}^2 \psi^n) - G(\cdot, \bar{u}_0, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) + \mathcal{G}^{\mathbb{P}}(\varphi - \psi^n) \right\}_{\tau_i^n} \right]. \end{aligned}$$

By Assumption 3.3, we may choose  $\mathbb{P}_n \in \mathcal{P}_{L_0}$  such that

$$\left\{ G(\cdot, \bar{u}_0, \partial_\omega \psi^n, \partial_{\omega\omega}^2 \psi^n) - G(\cdot, \bar{u}_0, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) + \mathcal{G}^{\mathbb{P}}(\varphi - \psi^n) \right\}_{\tau_i^n} = 0.$$

Then  $\delta_n \geq [\frac{c}{3} - C\eta] \mathbb{E}^{\mathbb{P}_n}[\mathbf{H}]$ . Set  $\eta := \frac{c}{6C}$ , we have  $\delta_n \geq \frac{c}{6} \underline{\mathcal{E}}^{L_0}[\mathbf{H}]$ . Sending  $n \rightarrow \infty$  we obtain  $\underline{\mathcal{E}}^{L_0}[\mathbf{H}] = 0$ . However, since  $\mathbf{H} \in \mathcal{H}$ , one can easily show that  $\underline{\mathcal{E}}^{L_0}[\mathbf{H}] > 0$ . This is a contradiction.  $\blacksquare$

Finally, we may complete

**Proof of Theorem 3.4 Existence.** Recall from Proposition 5.1 that  $u := \underline{u} = \bar{u}$ . Then, Propositions 6.1 and 6.2 imply immediately that  $u$  is a viscosity solution of PPDE (2.12).  $\blacksquare$

## 7 On Assumption 3.1

In this section we discuss the validity of our Assumption 3.1 which is clearly related to the classical Perron approach, the key-argument for the existence in the theory of viscosity solutions as shown by Ishii [10]. However, our definition of  $\bar{v}$  and  $\underline{v}$  involves classical supersolutions and subsolutions, while the classical definition in [10] involves viscosity solutions. The main issue here is to approximate viscosity solutions by classical supersolutions or subsolutions. This is a difficult question which requires some restrictions on the nonlinearity. In this section, we provide some sufficient conditions, and we hope to address this issue in a more systematic way in some future research.

For the ease of presentation, we first simplify the notations in Assumption 3.1. Let

$$\begin{aligned} O &:= \{x \in \mathbb{R}^d : |x| < 1\}, \quad \bar{O} := \{x \in \mathbb{R}^d : |x| \leq 1\}, \quad \partial O := \{x \in \mathbb{R}^d : |x| = 1\}; \\ \mathcal{O} &:= [0, T) \times O, \quad \bar{\mathcal{O}} := [0, T] \times \bar{O}, \quad \partial \mathcal{O} := ([0, T] \times \partial O) \cup (\{T\} \times O). \end{aligned} \tag{7.10}$$

We shall consider the following (deterministic) PDE on  $\mathcal{O}$ :

$$\mathbf{L}v := -\partial_t v - g(s, x, v, Dv, D^2v) = 0 \quad \text{in } \mathcal{O} \quad \text{and} \quad v = h \quad \text{on } \partial \mathcal{O}. \tag{7.11}$$

**Assumption 7.1** (i)  $g$  and  $h$  are continuous in  $(t, x)$ ,  
(ii)  $g$  is uniformly Lipschitz continuous in  $(y, z, \gamma)$ , and nondecreasing in  $\gamma$ ,  
(iii) The PDE (7.11) satisfies existence and comparison in the sense of viscosity solutions within the class of bounded functions.

More precisely, the last item (iii) states that for any bounded functions  $v^1, v^2$  satisfying  $\mathbf{L}v^1 \leq 0 \leq \mathbf{L}v^2$  on  $\mathcal{O}$ , in the sense of viscosity solutions, and  $v^1 \leq h \leq v^2$  on  $\partial\mathcal{O}$ , we have  $v^1 \leq v^2$ . Define

$$\begin{aligned}\bar{v}(t, x) &:= \inf \left\{ w(t, x) : w \text{ classical supersolution of PDE (7.11)} \right\}, \\ \underline{v}(t, x) &:= \sup \left\{ w(t, x) : w \text{ classical subsolution of PDE (7.11)} \right\}.\end{aligned}$$

By the comparison principle we have  $\underline{v} \leq v \leq \bar{v}$ , where  $v$  denotes the unique viscosity solution of PDE (7.11).

Our main result of this section is

**Proposition 7.2** *Under Assumption 7.1, we have  $\bar{v} = \underline{v}$  in the following three cases:*

- (i)  $g$  is convex in  $(y, z, \gamma)$ ,  $g_\delta(\cdot, \gamma) := \inf_{A \in \mathbb{S}_+^d} \{g(\cdot, \gamma + A) - \delta I_d : A\} > -\infty$  for  $0 \leq \delta \leq c_0$ , for some  $c_0 > 0$ , and  $g_\delta \rightarrow g$  as  $\delta \searrow 0$ ,
- (ii)  $g$  is convex in  $\gamma$  and uniformly elliptic: for some constant  $c_0 > 0$ ,

$$g(\cdot, \gamma_1) - g(\cdot, \gamma_2) \geq c_0 I_d : (\gamma^1 - \gamma^2) \quad \text{for any } \gamma^1 \geq \gamma^2.$$

- (iii)  $g$  is uniformly elliptic and  $d \leq 2$ .

**Proof** Without loss of generality we assume

$$g(\cdot, y_1, \cdot) - g(\cdot, y_2, \cdot) \leq y_2 - y_1 \quad \text{for all } y_1 \geq y_2. \quad (7.12)$$

- (i) We proceed in two steps.

*Step 1.* Let  $\mu$  be a smooth molifier on  $\mathcal{O}$ , and define  $\bar{w}^\delta := v * \mu_\delta + c_\delta$  for any  $\delta > 0$ , where  $c_\delta := \sup_{\partial\mathcal{O}} |v * \mu_\delta - h|$ . Then  $\bar{w}^\delta \in C^{1,2}(\bar{\mathcal{O}})$  and  $\bar{w}^\delta \geq h$  on  $\partial\mathcal{O}$ , and by the convexity argument of Krylov [11], it follows that  $\bar{w}^\delta$  satisfies in the viscosity sense:

$$\begin{aligned}-\partial_t \bar{w}^\delta - g(\cdot, \bar{w}^\delta, D\bar{w}^\delta, D^2\bar{w}^\delta) &\geq \left\{ -\partial_t v - g(\cdot, v + c_\delta, Dv, D^2v) \right\} * \mu_\delta \\ &\geq \left\{ -\partial_t v - g(\cdot, v, Dv, D^2v) \right\} * \mu_\delta = 0,\end{aligned}$$

where we used (7.12). This implies  $\bar{v} \leq \bar{w}^\delta$ . Note that  $\bar{w}^\delta - v \leq 2c_\delta$  on  $\partial\mathcal{O}$ . Following similar arguments as in Lemma 5.2 one can easily show that  $\bar{w}^\delta(0, \mathbf{0}) - v(0, \mathbf{0}) \leq Cc_\delta$ .

Clearly  $c_\delta \rightarrow 0$  as  $\delta \searrow 0$ , then  $\bar{v}(0, \mathbf{0}) = v(0, \mathbf{0})$ .

*Step 2.* Let  $0 < \delta < c_0$ . Then  $g_\delta \leq g$ , and for any  $\gamma' \in \mathbb{S}_+^d$ , we directly verify that

$$\begin{aligned} g_\delta(\cdot, \gamma) &= \inf_{A \geq -\gamma'} \{g(\cdot, \gamma + \gamma' + A) - \delta I_d : (A + \gamma')\} \\ &\leq \inf_{A \geq \mathbf{0}} \{g(\cdot, \gamma + \gamma' + A) - \delta I_d : (A + \gamma')\} = -\delta I_d : \gamma' + g_\delta(\cdot, \gamma + \gamma'), \end{aligned}$$

that is  $g_\delta$  is uniformly elliptic. Finally, consider some  $\lambda \in [0, 1]$ ,  $\gamma_0, \gamma_1 \in \mathbb{S}^d$ , and set  $\bar{\gamma} := (1 - \lambda)\gamma_0 + \lambda\gamma_1$ . For all  $A_0, A_1 \in \mathbb{S}_+^d$ , we have

$$\begin{aligned} g_\delta(\cdot, \bar{\gamma}) &\leq g(\cdot, \bar{\gamma} + (1 - \lambda)A_0 + \lambda A_1) - \delta I_d : ((1 - \lambda)A_0 + \lambda A_1) \\ &\leq (1 - \lambda)(g(\gamma_0 + A_0) - \delta I_d : A_0) + \lambda(g(\gamma_1 + A_1) - \delta I_d : A_1). \end{aligned}$$

By the arbitrariness of  $A_0, A_1$  in  $\mathbb{S}_+^d$ , this shows that  $g_\delta$  is convex in  $\gamma$ .

Next, let  $g_\delta^n$  and  $h^n$  be smooth mollifiers of  $g_\delta$  and  $h$  such that  $g_\delta^n \uparrow g_\delta$  and  $h^n \uparrow h$  uniformly. Note that  $g_\delta^n$  inherits the uniform ellipticity. Then we may apply Theorem 14.15 of Lieberman [12] to deduce the existence of a unique bounded classical solution  $\underline{w}^{n,\delta}$  of the equation

$$-\partial_t \underline{w}^{n,\delta} - g_\delta^n(\cdot, \underline{w}^{n,\delta}, D\underline{w}^{n,\delta}, D^2 \underline{w}^{n,\delta}) = 0 \quad \text{on } \mathcal{O}, \quad \text{and } \underline{w}^{n,\delta} = h^n \quad \text{on } \partial\mathcal{O}.$$

This implies that  $\mathbf{L}\underline{w}^{n,\delta} \leq 0$  in  $\mathcal{O}$  and  $\underline{w}^{n,\delta} \leq h$  on  $\partial\mathcal{O}$ , and thus  $\underline{w}^{n,\delta}(0, \mathbf{0}) \leq \underline{v}(0, \mathbf{0})$ . By the comparison and the stability of viscosity solutions, it follows that  $\underline{w}^{n,\delta} \uparrow \underline{w}^\delta$ , where  $\underline{w}^\delta$  is the unique viscosity solution of

$$-\partial_t \underline{w}^\delta - g_\delta(\cdot, \underline{w}^\delta, D\underline{w}^\delta, D^2 \underline{w}^\delta) = 0 \quad \text{on } \mathcal{O}, \quad \text{and } \underline{w}^\delta = h \quad \text{on } \partial\mathcal{O}.$$

Note that  $g_\delta \uparrow g$  as  $\delta \downarrow 0$ . Then by the comparison and then stability of viscosity solutions again we see that  $\underline{w}^\delta \uparrow v$ . This implies that  $\underline{v}(0, \mathbf{0}) = v(0, \mathbf{0})$ .

(ii) For any  $\alpha > 0$ , we define  $O^\alpha := \{x \in \mathbb{R}^d : |x| < 1 + \alpha\}$ ,  $\mathcal{O}^\delta := [0, (1 + \alpha)T) \times O^\alpha$ , and similar to (7.10), define their closures and boundaries. Let  $\mu, \eta$  be smooth mollifiers on  $\mathcal{O}$  and  $\mathcal{O}^1 \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$  and define: for any  $\alpha' > 0$ ,

$$\begin{aligned} h_\alpha(t, x) &:= (h * \mu_\alpha)\left(\frac{t}{1+\alpha}, \frac{x}{1+\alpha}\right), \quad (t, x) \in \bar{\mathcal{O}}^\alpha, \\ g_0(t, x, y, z, \gamma) &:= \min_{(t', x') \in \mathcal{O}} \{g(t', x', y, z, \gamma) + 2\rho_0(|t - t'| + |x - x'|)\} \\ g_{\alpha'} &:= (g_0 * \eta_{\alpha'}), \quad (t, x, y, z, \gamma) \in \mathcal{O}^1 \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d. \end{aligned}$$

By the uniform continuity of  $g$ , we have  $c(\alpha') := \|g - g_{\alpha'}\|_\infty \rightarrow 0$  as  $\alpha' \searrow 0$ . Set

$$\underline{g}_{\alpha'} := g_{\alpha'} - c(\alpha'), \quad \text{and} \quad \bar{g}_{\alpha'} := g_{\alpha'} + c(\alpha').$$

By our assumptions on  $g$  and  $h$ , it follows from Theorem 14.15 of Lieberman [12] that there exist  $\underline{v}_{\alpha,\alpha'}, \bar{v}_{\alpha,\alpha'} \in C^{1,2}(\mathcal{O}^\alpha) \cap C(\bar{\mathcal{O}}^\alpha)$  solutions of the equations:

$$\begin{aligned} (\underline{E}_{\alpha,\alpha'}) : -\partial_t v - \underline{g}_{\alpha'}(., v, Dv, D^2v) &= 0 \text{ in } \mathcal{O}^\alpha, \text{ and } v = h_\alpha \text{ on } \partial\mathcal{O}^\alpha \\ (\bar{E}_{\alpha,\alpha'}) : -\partial_t v - \bar{g}_{\alpha'}(., v, Dv, D^2v) &= 0 \text{ in } \mathcal{O}^\alpha, \text{ and } v = h_\alpha \text{ on } \partial\mathcal{O}^\alpha, \end{aligned}$$

respectively. In particular, their restriction to  $\bar{\mathcal{O}}$  are in  $C^{1,2}(\bar{\mathcal{O}})$ . By comparison principle,  $\underline{v}_{\alpha,\alpha'} \leq \bar{v}_{\alpha,\alpha'}$ . Moreover, it follows from (7.12) that:

$$\bar{g}_{\alpha'}(., y + 2c(\alpha'), .) \leq \bar{g}_{\alpha'}(., y, .) - 2c(\alpha') = \underline{g}_{\alpha'}(., y, .).$$

This shows that  $v_{\alpha,\alpha'} + 2c(\alpha')$  is a classical supersolution of  $(\bar{E}_{\alpha,\alpha'})$ , and therefore

$$\underline{v}_{\alpha,\alpha'} + 2c(\alpha') \geq \bar{v}_{\alpha,\alpha'} \geq \underline{v}_{\alpha,\alpha'}.$$

Additionally, notice that the solutions  $\underline{v}_{\alpha,\alpha'}, \bar{v}_{\alpha,\alpha'}$  are bounded uniformly in  $\alpha, \alpha'$  for  $\alpha, \alpha'$  small enough. The generators  $\underline{g}_{\alpha'}, \bar{g}_{\alpha'}$  have the same uniform ellipticity constants as  $g$ , and they verify the hypothesis of Theorem 14.13 of Lieberman [12] uniformly in  $\alpha'$ . Therefore  $\underline{v}_{\alpha,\alpha'}, \bar{v}_{\alpha,\alpha'}$  are Lipschitz continuous with the same Lipschitz constant for all  $\alpha, \alpha'$ . Then, denoting  $\bar{h}_{\alpha,\alpha'} := \bar{v}_{\alpha,\alpha'}|_{\partial\mathcal{O}}$  and  $\underline{h}_{\alpha,\alpha'} := \underline{v}_{\alpha,\alpha'}|_{\partial\mathcal{O}}$ , this implies that

$$c(\alpha, \alpha') := \max \{ \|\bar{h}_{\alpha,\alpha'} - h\|_\infty, \|\underline{h}_{\alpha,\alpha'} - h\|_\infty \} \longrightarrow 0, \text{ as } \alpha \rightarrow 0, \text{ uniformly in } \alpha'.$$

Now for fixed  $\varepsilon > 0$ , choose  $\alpha_0, \alpha'_0 > 0$  so that  $c(\alpha_0, \alpha') < \varepsilon/4$  for all  $\alpha' > 0$ , and  $c(\alpha'_0) \leq \varepsilon/4$ . Then,  $\bar{w}_{\alpha_0, \alpha'_0} := \bar{v}_{\alpha_0, \alpha'_0} + c(\alpha_0, \alpha'_0)$  and  $\underline{w}_{\alpha_0, \alpha'_0} := \underline{v}_{\alpha_0, \alpha'_0} - c(\alpha_0, \alpha'_0)$  are respectively classical supersolution and subsolution of (7.11) on  $\bar{\mathcal{O}}$ . Thus  $\underline{w}_{\alpha_0, \alpha'_0} \leq \underline{v}$  and  $\bar{w}_{\alpha_0, \alpha'_0} \geq \bar{v}$ . Therefore,

$$\bar{v} - \underline{v} \leq \bar{w}_{\alpha_0, \alpha'_0} - \underline{w}_{\alpha_0, \alpha'_0} = \bar{v}_{\alpha_0, \alpha'_0} - \underline{v}_{\alpha_0, \alpha'_0} + 2c(\alpha_0, \alpha'_0) \leq 2c(\alpha'_0) + 2c(\alpha_0, \alpha'_0) \leq \varepsilon.$$

Then it follows from the arbitrariness of  $\varepsilon$  that  $\bar{v} = \underline{v}$ .

(iii) We refer to Pham and Zhang [19] for the proof in this case. ■

**Remark 7.3** (i) Proposition 7.2 (i) allows to deal with degenerate PPDEs. For example, in the case of degenerate  $G$ -expectation with  $G = G(\gamma) = \frac{1}{2} \sup_{0 \leq \sigma \leq \bar{\sigma}} [\sigma^2 \gamma]$ , one can compute straightforwardly that  $g(\gamma) = \frac{1}{2} \bar{\sigma}^2 \gamma^+$  and  $g_\delta(\gamma) = \frac{1}{2} [\bar{\sigma}^2 \gamma^+ - \delta \gamma^-]$ , and thus satisfy the conditions in Proposition 7.2 (i).

(ii) Proposition 7.2 (iii) does not require convexity of  $G$  in  $\gamma$ , and thus allows us to deal with path dependent Bellman-Isaacs equations. See Pham and Zhang [19] for its application in stochastic differential games. ■

## References

- [1] Cheridito, P., Soner, H.M. and Touzi, N., Victoir, N. (2007) *Second order BSDE's and fully nonlinear PDE's*, *Communications in Pure and Applied Mathematics*, 60, 1081-1110.
- [2] Crandall, M.G. and Lions, P.-L. (1983), *Viscosity solutions of Hamilton-Jacobi equations*, *Transactions of the American Mathematical Society* 277 (1): 1-42.
- [3] Crandall, M.G., Ishii, H., and Lions, P.-L. (1992) *User's guide to viscosity solutions of second order partial differential equations*, *Bull. Amer. Math. Soc. (NS)*, **27**, 1-67.
- [4] Dupire, B. (2009) *Functional Itô calculus*, papers.ssrn.com.
- [5] Ekren, I., Keller, C., Touzi, N., and Zhang, J. *On Viscosity Solutions of Path Dependent PDEs*, *Annals of Probability*, to appear.
- [6] Ekren, I., Touzi, N., and Zhang, J. *Optimal Stopping under Nonlinear Expectation*, preprint.
- [7] Ekren, I., Touzi, N., and Zhang, J. *Viscosity Solutions of Fully Nonlinear Parabolic Path Dependent PDEs: Part I*, preprint.
- [8] Fleming, W. and Soner, H.M. (2006) *Controlled Markov Processes and Viscosity Solutions*, 2nd ed., Springer, New York.
- [9] Hu, M., Ji, S., Peng, S., and Song, Y. (2012) *Backward Stochastic Differential Equations Driven by G-Brownian Motion*, arXiv:1206.5889.
- [10] Ishii, H. (1987), *Perron's method for Hamilton-Jacobi equations*, *Duke Mathematical Journal*, 55, no. 2, 369-384.
- [11] Krylov, N.V. (2000), *On the rate of convergence of finite-difference approximations for Bellman's equations with variable coefficients*. *Probability Theory and Related Fields*, 117, 1-16.
- [12] Lieberman, G. M. *Second order parabolic differential equations*, World Scientific, 1998.
- [13] Lukyanov, N. (2007) *On viscosity solution of functional Hamilton-Jacobi type equations for hereditary systems*, *Proceedings of the Steklov Institute of Mathematics*, Suppl. 2, 190-200.



- [14] Ma, J., Yin, H., and Zhang, J. (2012) *On Non-Markovian Forward Backward SDEs and Backward Stochastic PDEs*, *Stochastic Processes and Their Applications*, to appear.
- [15] Oksendal, B., Sulem, A. and Zhang, T. (2011) *Singular control of SPDEs and backward SPDEs with reflection*, Preprint.
- [16] E. Pardoux and S. Peng (1990) *Adapted solutions of backward stochastic differential equations*, *System and Control Letters*, **14**, 55-61.
- [17] Peng, S. (2010) *Backward stochastic differential equation, nonlinear expectation and their applications*, *Proceedings of the International Congress of Mathematicians*, Hyderabad, India.
- [18] Peng, S. (2011) *Note on Viscosity Solution of Path-Dependent PDE and G-Martingales*, arXiv:1106.1144.
- [19] Pham, T. and Zhang, J. (2012) *Two Person Zero-sum Game in Weak Formulation and Path Dependent Bellman-Isaacs Equation*, preprint.
- [20] Soner, H. M. Touzi, N. and Zhang, J. (2012) *Well-posedness of second order backward SDEs*, *Probability Theory and Related Fields*, 153, 149-190.